§7.6 估计的渐近分布

- 定义6.2. 设$T_n = T_n(X_1, \cdots, X_n)$ 满足:

$$\sqrt{n}(T_n - g(\theta)) \xrightarrow{d} Z \sim N(0, \sigma^2), \quad \forall \theta \in \Theta,$$

则称T_n 是渐近正态的, 其中$\sigma^2 = \sigma_\theta^2$ 称为渐进方差.

- 工具: CLT & Δ 方法.
- 定理6.3 (Δ 方法). 设$\sqrt{n}(T_n - \theta) \xrightarrow{d} Z \sim N(0, \tau^2)$, $h'(\theta)$ 存在且不为0, 则

$$\sqrt{n}\big(h(T_n) - h(\theta)\big) \xrightarrow{d} W \sim N(0, h'(\theta)^2\tau^2).$$

例6.1. 总体: $X \sim N(\mu, \sigma^2)$, 样本量: n.

- UMVU 估计: $\hat{\mu} = \bar{X}$, $\widehat{\sigma^2} = S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$.
- $\hat{\mu}$ 渐近正态: 事实上,

$$\sqrt{n}(\bar{X} - \mu) \sim N(0, \sigma^2).$$

- 定理7.1. $(n-1)S^2 \stackrel{d}{=} \sigma^2 K_{n-1}$, 其中, $K_{n-1} \sim \chi^2(n-1)$.
- S^2 渐近正态: CLT,

$$\frac{\sum_{i=1}^{n-1} Z_i^2 - (n-1)}{\sqrt{n-1}} \xrightarrow{d} W \sim N(0, \mathrm{var}(Z^2))$$

$$\Rightarrow \sqrt{n}(S^2 - \sigma^2) \approx \sqrt{n-1}(S^2 - \sigma^2) \xrightarrow{d} \sigma^2 W \sim N(0, 2\sigma^4).$$

例6.3. 总体: $X \sim N(\mu, 1)$, 待估量: $g(\mu) = P_\mu(X \leqslant x_0)$.

- 方法一、$g(\mu) = P_\mu(X - \mu \leqslant x_0 - \mu) = \Phi(x_0 - \mu)$.
- 由CLT, μ 的最大似然估计$\hat{\mu} = \bar{X}$ 渐近正态, 渐近方差$= 1$.
- 再由Δ 方法, $g(\mu)$ 的最大似然估计$g(\hat{\mu}) = \Phi(x_0 - \bar{X})$ 渐近正态, 渐近方差为

$$\sigma_1^2 = g'(\mu)^2 \cdot 1 = \varphi(x_0 - \mu)^2.$$

- 方法二、$\frac{1}{n}\sum_{i=1}^n 1_{\{X_i \leqslant x_0\}} \xrightarrow{P_\mu} P_\mu(X_i \leqslant x_0) = g(\mu)$, 渐近正态, 渐近方差为

$$\sigma_2^2 = \mathrm{var}(1_{\{X \leqslant x_0\}}) = g(\mu)(1 - g(\mu)) = \Phi(x_0 - \mu)\big(1 - \Phi(x_0 - \mu)\big).$$

- 习题七、30: $\sigma_1^2 \leqslant \sigma_2^2$.

例6.3 (续). 总体: $X \sim N(\mu, 1)$, 待估量: $g(\mu) = \Phi(x_0 - \mu)$.

- $\hat{\mu} = \bar{X}$ 是完全充分统计量, 但$g(\hat{\mu})$ 不是$g(\mu)$ 的无偏估计.
- 令$h(\mu) = \Phi\left(\sqrt{\frac{n}{n-1}}(x_0 - \mu)\right)$.
- 记$p_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, 则

$$
\begin{aligned}
E_\mu h(\hat{\mu}) &= E_\mu \Phi\left(\sqrt{\frac{n}{n-1}}(x_0 - \bar{X})\right) \\
&= \int_{-\infty}^{\infty} p_{\mu,\frac{1}{n}}(y) \Phi\left(\sqrt{\frac{n}{n-1}}(x_0 - y)\right) dy \\
&= \int_{-\infty}^{\infty} p_{\mu,\frac{1}{n}}(y) \int_{-\infty}^{\sqrt{\frac{n}{n-1}}(x_0-y)} p_{0,1}(z) dz dy \\
&= P\left(Z \leqslant \sqrt{\frac{n}{n-1}}(x_0 - Y)\right),
\end{aligned}
$$

其中, Y, Z 相互独立, $Y \sim N(\mu, \frac{1}{n})$, $Z \sim N(0, 1)$.

例6.3 (续). 总体: $X \sim N(\mu, 1)$, 待估量: $g(\mu) = \Phi(x_0 - \mu)$.

- 已有: 取Y, Z 相互独立, $Y \sim N(\mu, \frac{1}{n})$, $Z \sim N(0, 1)$, 则

$$E_\mu h(\hat{\mu}) = P\left(Z \leqslant \sqrt{\frac{n}{n-1}}(x_0 - Y)\right),$$

- $\sqrt{\frac{n-1}{n}}Z + Y \sim N(\mu, 1)$. 因此,

$$E_\mu h(\hat{\mu}) = P\left(\sqrt{\frac{n-1}{n}}Z + Y - \mu \leqslant x_0 - \mu\right) = \Phi(x_0 - \mu).$$

- $\hat{\mu}$ 是完全充分统计量, $h(\hat{\mu})$ 是$g(\mu)$ 的无偏估计, 因此是UMVU 估计.