Characteristic Class and epsilon factor of an étale sheaf

Joint with Umezaki and Zhao.

"hǎi-vier"

Thank Prof Javier Fresán very much for his invitation, also thank the organizers.

When I say "CC", I mean characteristic cycle

"SS", I mean singular support. for short

smaller "cc", I mean characteristic class.

<u>Notation</u> Let k be a finite field of characteristic $p>0$

Let $\ell \neq p$ be a prime

Let Λ be a finite extension of $\mathbb{F}_\ell$ or $\mathbb{Q}_\ell$.

All schemes$/k$ are assumed to be separated of finite type over k, except for local trait

X/k smooth projective, purely of dimension d.

$\mathcal{F} \in D_c^b(X_{et}, \Lambda)$

<u>Goal</u> Prove a <u>twist formula</u> for <u>global epsilon factor</u>, and apply it to study the push-forward of <u>characteristic class</u>
Swan class

§1 Introduction

① The <u>global epsilon factor</u> of $\mathcal{F}$ is defined to be $\varepsilon(X,\mathcal{F}) = \det(-\mathrm{Frob}_k; R\Gamma(X_{\bar{k}}, \mathcal{F}$

where Frob_k is the geometric Frobenius, i.e., the inverse of the Frobenius substitution $x \mapsto x^{\#k}$ on $\bar{k}$.

② The <u>characteristic class of $\mathcal{F}$</u> is the zero cycle class

$$cc_X \mathcal{F} = \langle CC\mathcal{F}, T^*_X X \rangle_{T^*X} \in CH_0(X).$$

③ By Kato & Saito's unramified class field theory for X, we have a reciprocity map

$$CH_0(X) \xrightarrow{\rho_X} \pi_1^{ab}(X)$$
$$[s] \longmapsto [\mathrm{Frob}_s] \text{ geo. Frob.}$$

Even though, Frob_s is defined up to conjugation, but in $\pi_1^{ab}(X)$, it is well defined.

Which is injective with dense image.

The following global twist formula was conjectured by Kato-T.Saito around ~~2004~~ 2009?

Theorem A (Umezaki-Y-Z, 2017)

For any smooth sheaf $\mathcal{G} \in D_c^b(X,\Lambda)$, we have

$$\varepsilon(X, \mathcal{F}\otimes\mathcal{G}) = \varepsilon(X,\mathcal{F})^{\mathrm{rk}\mathcal{G}} \cdot \det\mathcal{G}(-\rho_X(cc_X\mathcal{F}))$$

$$\underset{cc_X\mathcal{F}}{CH_0(X)} \xrightarrow{\rho_X} \pi_1^{ab}(X) \xrightarrow{\det\mathcal{G}} \Lambda^\times.$$

Remark In Kato-T.Saito's paper, they defined the Swan class $Sw^{ks}(\mathcal{F}) \in CH_0(X\setminus U)\otimes\mathbb{Q}$ for a smooth sheaf $\mathcal{F} \in D_c^b(U,\Lambda)$ on $U \subseteq X$ by using alteration and logarithmic blow-up. The global twist formula was written in terms of $Sw^{ks}(\mathcal{F})$ in that time.

Conjecture B (T. Saito, 2016, weak form) For any smooth sheaf $\mathcal{F}$ on $U \overset{j}{\hookrightarrow} X$, ($k$: any perfect field.)
we have $$Sw^{ks}(\mathcal{F}) = Sw^{cc}(\mathcal{F})$$
where $Sw^{cc}\mathcal{F} = \langle T^*_X X, \mathrm{rank}\mathcal{F}\cdot CC\, j_!\Lambda - CC j_!\mathcal{F}\rangle_{T^*X}$ in $CH_0(X\setminus U)$.

— Both Sw^{ks} and Sw^{cc} are satisfies the higher GOS formula:

$$\chi_c(U_{\bar{k}}, \mathcal{F}) = \mathrm{rank}\mathcal{F}\cdot\chi_c(U_{\bar{k}},\mathcal{F}) - \deg Sw\,\mathcal{F}.$$

We proved Conjecture B if X is a proj smooth surface over a finite field
or if we ~~assume~~ assume resolution of singularities and assume
proper push-forward of Sw^{cc} (or CC or $cc_X\mathcal{F}$) by generically
finite and surjective map.

I will back to this part if I have more time.

Some known results of twist formula

1) local twist formula, (Deligne + Henniart 1981), globalization?

2) If $\mathcal{F}$ is smooth (has no ramification),
$\boxed{\text{1984, S. Saito, proved an explicit formula for } \varepsilon(X,\mathcal{F})}$

1984 Henniart : explicit formula for local epsilon factor modulo roots of unity of p-power order.

4) 2016, Tomoyuki Abe and Deepam Patel, twist formula for de Rham epsilon factor (~~asy~~ via microlocal geometry.

It's still open for microlocal description of $\begin{matrix}\det R\Gamma \\ SS \\ CC\end{matrix}$ for ℓ-adic cohomology.

Application of twist formula (proper push-forward)

~~total~~ characteristic class

Let $K(X,\Lambda)$ be the Grothendieck group of $D_c^b(X,\Lambda)$,

$$\begin{array}{ccccc} K(X,\Lambda) & \longrightarrow & CH_d(\mathbb{P}(T^*X\oplus 1)) & \xrightarrow{\simeq} & CH_\bullet(X) = \bigoplus_{i=1}^{d} CH_i(X) \\ \mathcal{F} & \longmapsto & \overline{CC\mathcal{F}} = \overline{\mathbb{P}(CC\mathcal{F}\oplus 1)} & \longmapsto & \pi_*(c(\xi)\cap \overline{CC\mathcal{F}}) \end{array}$$

$d = \dim X$

We have
$cc_{X,0}\mathcal{F} = cc_X\mathcal{F}$
$cc_{X,d}\mathcal{F} = (-1)^d \operatorname{rank}\mathcal{F}$ generic rank
$cc_{X,d-1}\mathcal{F}$ = artin divisor = ~~total~~ dimension divisor

$$\begin{array}{c} T^*X \hookrightarrow \mathbb{P}(T^*X\oplus 1) \\ \downarrow \pi \\ X \end{array}$$

where ξ is the universal quotient bundle of rank d on $\pi^*(T^*X\oplus 1)$.

If $k=\mathbb{C}$, by a theorem of V. Ginsburg, the following diagram is commutative for any projective morphism $f: X \longrightarrow Y$ in $\underline{Sm/k}$

$$\begin{array}{ccc} K(X,\Lambda) & \xrightarrow{cc_{X,\bullet}} & CH_\bullet(X) \\ \downarrow f_* & (**) & \downarrow f_* \\ K(Y,\Lambda) & \xrightarrow{cc_{Y,\bullet}} & CH_\bullet(Y) \end{array}$$

except for the degree zero part.

But in char >0, $(**)$ is not commutative by a philosophy of Grothendieck.

Counter example The Frobenius map $X=\mathbb{P}^n \xrightarrow{F} X$ is radicial and surjective.

$CH_0(\mathbb{P}^n) = \mathbb{Z}$ $\xrightarrow{F_*}$ $CH_0(\mathbb{P}^n)$ multiplication by ...

Corollary C Let $f: X \longrightarrow Y$ be a projective map between smooth schemes over a finite field k. For any $\mathcal{F} \in D_c^b(X, \Lambda)$, we have

$$f_* cc_X \mathcal{F} = cc_Y Rf_* \mathcal{F} \quad \text{in } CH_0(Y).$$

直接设为 $\Lambda = \overline{\mathbb{Q}}_\ell$

proof $\Lambda = \overline{\mathbb{Q}}_\ell$.

For any χ: 1-rep. of $\pi_1^{ab}(Y)$

$$\begin{tikzcd} CH_0(X) \arrow[r, "\rho_X"] \arrow[d, "f_*"] & \pi_1^{ab}(X) \arrow[d] & \\ CH_0(Y) \arrow[r, "\rho_Y"'] & \pi_1^{ab}(Y) \arrow[r, "\chi"] & \overline{\mathbb{Q}}_\ell^\times \end{tikzcd}$$

$$\chi(\rho_Y(-cc_Y Rf_* \mathcal{F})) = \frac{\varepsilon(Y, Rf_*\mathcal{F} \otimes \chi)}{\varepsilon(Y, Rf_* \mathcal{F})}$$

$$= \frac{\varepsilon(X, \mathcal{F} \otimes f^*\chi)}{\varepsilon(X, \mathcal{F})} = (f^*\chi)(\rho_X(-cc_X \mathcal{F}))$$

$$= \chi(\rho_Y(f_* cc_X \mathcal{F})).$$

$\overline{\mathbb{Q}}^\times \supseteq \mathbb{Q}/\mathbb{Z}$ roots of unity.

injective of $\rho_Y \Rightarrow f_* cc_X \mathcal{F} = cc_Y Rf_* \mathcal{F}$. $\boxtimes$.

We start to prove the curve case of twist formula.

§2 Local epsilon factor and Deligne-Laumon's product formula

$X \xrightarrow{f} \operatorname{Spec} k$ smooth projective curve over a finite field k.

$\mathcal{F} \in D_c^b(X, \overline{\mathbb{Q}}_\ell)$

Grothendieck L-function $L(X, \mathcal{F}, t) = \det(1 - t \cdot \mathrm{Frob}_k; R\Gamma(X_{\bar{k}}, \mathcal{F}))^{-1}$

$$= \prod_{x \in |X|} \frac{1}{\det(1 - t^{\deg(x)} \mathrm{Frob}_x; \mathcal{F}_{\bar{x}})}$$

It satisfies the following functional equation:

$q = \# k$

$L(X, \mathcal{F}(n); t) = L(X, \mathcal{F}; q^{-n} t)$

$$L(X, \mathcal{F}, t) = t^{-\chi(X_{\bar{k}}, \mathcal{F})} \cdot \varepsilon(X, \mathcal{F}) \cdot \underbrace{L(X, D(\mathcal{F}), t^{-1})}_{L(X, \mathcal{F}^\vee, q^{-1}t^{-1})}$$

$\mathcal{F}^\vee = R\mathcal{H}om(\mathcal{F}, \overline{\mathbb{Q}}_\ell)$

where $D\mathcal{F} = R\mathcal{H}om(\mathcal{F}, Rf^! \overline{\mathbb{Q}}_\ell)$ is the dual of $\mathcal{F}$.

Langlands: L-function of auto. forms $=$ products of local L-functions.

ε-factor $=$ products of certain local ε-factors.

Theorem (Langlands 1970, Deligne 1972).

Let ψ be a fixed non-trivial additive character of k.

There exists a unique map $\varepsilon_\psi := \varepsilon$:

$$\text{triples } (T, \mathcal{F}, \omega) \longmapsto \varepsilon(T, \mathcal{F}, \omega) \in \overline{\mathbb{Q}}_\ell^\times$$

where T is a henselian trait $\underbrace{}_{\text{of } \cancel{\text{equal}} \text{ equi-char } p}$ with closed point s, generical point η, $k(s) \supseteq k$.

$\mathcal{F} \in D_c^b(T, \overline{\mathbb{Q}}_\ell)$

$\omega \in \Omega^1_{k(\eta)} \setminus \{0\}$.

Satisfying ① $\varepsilon(T, \mathcal{F}, \omega)$ depends only on the isom classes of the triple $(T, \mathcal{F}, \omega)$

② additive in $\mathcal{F}$

③ Induction formula for virtual sheaf of rank 0

$$\begin{array}{ccc} \eta_1 & & T_1 \\ \big\downarrow \text{finite separable ext.} & & \big\downarrow f \text{ normalization in } \eta_1 \\ \eta & & T \end{array}$$

$\mathcal{F}_1 \in D_c^b(T_1, \overline{\mathbb{Q}}_\ell)$

If $\operatorname{rank}(\mathcal{F}_1)_\eta = 0$, then

$$\varepsilon(T, f_*\mathcal{F}_1, \omega) = \varepsilon(T_1, \mathcal{F}_1, f^*\omega)$$

④ If $\mathcal{G}$ is smooth sheaf of rank 1 on η, which induce a character

$\chi : K^\times \longrightarrow \overline{\mathbb{Q}}^\times$ via the reciprocity homomorphism $r_K : K^\times \longrightarrow \operatorname{Gal}(\overline{K}/K)^{ab}$

其中 K = completion of $k(\eta)$ w.r.t valuation

uniformizer $\longmapsto$ Frob.

then $\varepsilon(T, j_*\mathcal{G}, \omega) = \operatorname{Tate}(\chi, \underset{\text{与}\psi\text{有关}}{\underset{\uparrow}{\Psi}})$

$j : \eta \to T$

$\Psi : K \longrightarrow \overline{\mathbb{Q}}_\ell^\times$

$\Psi(a) = \psi(\operatorname{Tr}_{k(s)|k}(\operatorname{res}_s(a\omega)))$.

<u>tate local constant</u>

$$\operatorname{Tate}(\chi, \Psi) = \begin{cases} \chi(\pi^{\operatorname{ord}_s(\omega)})\, q_s^{\operatorname{ord}_s(\omega)} \quad q_s = \#k(s) & \text{if } \chi|_{\mathcal{O}_K^\times} = 1 \\ \displaystyle\int_{\gamma^{-1}\mathcal{O}_K} \chi^{-1}(z)\Psi(z)\,dz & \text{if } \chi|_{\mathcal{O}_K^\times} \neq 1 \end{cases}$$

where $\gamma \in K^\times$ is an arbitrary element of valuation $a(\chi) + \operatorname{ord}_s(\omega)$, and $\int_{\mathcal{O}_K} dz = 1$.

$$a(\chi) = \begin{cases} 0 & \text{if } \chi \text{ is unramified} \\ [illegible] \end{cases}$$

① relation with det(Frob)

If $\mathcal{F}$ is supported on s, i.e., $\mathcal{F}_{\bar\eta}=0$, then $\varepsilon(T,\mathcal{F},\omega)=\det(-\mathrm{Frob}_s;\mathcal{F})^{-1}$.

Local twist formula (baby case of Deligne and Henniart)

If $\mathcal{G}$ is a smooth $\overline{\mathbb{Q}}_\ell$-sheaf on T, then we have

$$\varepsilon(T,\mathcal{F}\otimes\mathcal{G},\omega)=\varepsilon(T,\mathcal{F},\omega)^{\mathrm{rk}\,\mathcal{G}}\cdot\det(\mathrm{Frob}_s;\mathcal{G})^{a(T,\mathcal{F},\omega)}$$

alternative sum of generic rank

where $a(T,\mathcal{F},\omega)=a(T,\mathcal{F})+\mathrm{rank}\,\mathcal{F}_{\bar\eta}\cdot\mathrm{ord}_s(\omega)$

$=\mathrm{rank}\,\mathcal{F}_{\bar\eta}-\mathrm{rank}\,\mathcal{F}_{\bar s}+\mathrm{Sw}\,\mathcal{F}_{\bar\eta}+\mathrm{rank}\,\mathcal{F}_{\bar\eta}\cdot\mathrm{ord}_s(\omega)$ is the local Artin conductor.

Deligne-Laumon's product formula

smooth proj curve X/k, $\mathcal{F}$, ω: non-zero meromorphic 1-form on X.

$$\varepsilon(X,\mathcal{F})=q^{c(1-g)\,\mathrm{rank}\,\mathcal{F}}\prod_{x\in|X|}\underbrace{\varepsilon_x(\mathcal{F})}_{=\ \varepsilon(X_{(x)},\,\mathcal{F}|_{X_{(x)}},\,\omega|_{X_{(x)}})}$$

where $c=$ number of connected component $X\otimes\bar k$,

$g=$ genus of one of them. (or $\deg\omega=2g-2$)

Now we show the following

Local twist formula
product formula
GOS formula
$\Rightarrow$ global twist formula if $\dim X=1$.

$$\frac{\varepsilon(X,\mathcal{F}\otimes\mathcal{G})}{\varepsilon(X,\mathcal{F})^{\mathrm{rk}\,\mathcal{G}}}\overset{\text{product}}{=}\prod_{x\in|X|}\frac{\varepsilon_x(\mathcal{F}\otimes\mathcal{G})}{\varepsilon_x(\mathcal{F})^{\mathrm{rk}\,\mathcal{G}}}\ \overset{\text{local twist}}{=\!=}\ \prod_{x\in|X|}\det\mathcal{G}(\mathrm{Frob}_x)^{a_x(\mathcal{F},\omega)}$$

$$=\det\mathcal{G}\Big(\rho_X\Big(\sum_{x\in|X|}a_x(\mathcal{F},\omega)\cdot[x]\Big)\Big)$$

$$\overset{\text{GOS-formula}}{=}\det\mathcal{G}(\rho_X(-cc_X\mathcal{F}))$$

$cc_X\mathcal{F}=\langle CC(\mathcal{F}),T^*_XX\rangle$ $\Rightarrow$ $cc_X\mathcal{F}=-\sum_{x\in|X|}a_x(\mathcal{F},\omega)\cdot[x]$.

If $\dim X > 1$, we prove twist formula by induction on the dimension of X and choose good pencil for X. For this we need an induction formula for $cc_X\mathcal{F}$.

§3 Good fibration and induction formula for $cc_X\mathcal{F}$

Let $X \xrightarrow{f} Y$ be a flat morphism between projective smooth schemes

$\dim Y = 1$, ω: meromorphic 1-form

$C \subseteq T^*X$ conical closed subset.

We say that f is a <u>good fibration with respect to C and ω</u> if the following conditions are satisfied:

① f is C-transversal on $X \setminus \{u_1, \dots, u_m\}$ ← finite set of closed points of X

② if $i \neq j$, $f(u_i) \neq f(u_j)$, each fiber has at most one isolated char point w.r.t. C.

③ u_i and $f(u_i)$ have same residue field (lift the $\mathrm{Frob}_{f(u_i)}$ on C to Frob_{u_i} on X)

④ ω has neither poles nor zeros at $\Sigma = \{f(u_1), \dots, f(u_m)\}$

⑤ For all $v \in |Y|$, if $\mathrm{ord}_v(\omega) \neq 0$, then X_v is smooth and $X_v \xrightarrow{i_v} X$ is <u>properly</u> C-transversal. (then we can apply pull back formula for CC)

<u>Remark</u> If $k = \mathbb{F}_p^{alg}$, $X \subseteq \mathbb{P}^n \subseteq \mathbb{P}^N = \mathbb{P}$ ← r-fold Veronese embedding for $r \geq 3$ $\quad \mathbb{P}^n \longrightarrow \mathbb{P}^{\binom{n+d}{d}-1}$, $(a_0 : \cdots : a_n) \mapsto (a_0^d : a_0^{d-1}a_1 : \cdots : a_n^d)$

Saito-Yatagawa $\Rightarrow$ $\exists$ good pencil $L \subseteq \mathbb{P}^\vee$ of X such that

A_L meets X transversally

$X \cap A_L \to X$ properly C-tran.

$$X \longleftarrow X_L \longrightarrow L \quad \text{← good pencil}$$

$X_L \to X$ is properly C-tran.

$A_L \cap X$ is away from $\{u_1, \dots, u_m\}$.

If k = finite field, one need to take a finite extension $\bar{k}/k$.

Key Lemma $\mathcal{F} \in D_c^b(X, \Lambda)$. If $f: X \to Y$ is a good fibration associated to $(SS\mathcal{F}, \omega)$, then we have

$$cc_X\mathcal{F} = -\sum_{i=1}^{m} \mathrm{dimtot}\, R\Phi_{u_i}(\mathcal{F}, f)\cdot[u_i] - \sum_{v \in |Y| \setminus \Sigma} \mathrm{ord}_v(\omega)\cdot cc_{X_v}(\mathcal{F}|_{X_v}).$$

<u>Remark</u> We can use key Lemma to give an inductive construction of $cc_X\mathcal{F}$ starting from GOS formula for curves.

Under the assumptions in ~~the~~ Lemma | Recall $\mathcal{G}$ smooth on X.

If thmA is true for X_v for all $\mathrm{ord}_v(\omega)\neq 0$, then thm A is true for X.

$$\frac{\varepsilon(X,\mathcal{F}\otimes\mathcal{G})}{\varepsilon(X,\mathcal{F})^{\mathrm{rk}\mathcal{G}}} = \prod_{v\in|Y|} \frac{\varepsilon_v(Rf_*(\mathcal{F}\otimes\mathcal{G}))}{\varepsilon_v(Rf_*\mathcal{F})^{\mathrm{rk}\mathcal{G}}}$$

- local twist formula
- f-$\mathcal{C}$-tran over v
  $\Rightarrow$ f is universally locally acyclic w.r.t $\mathcal{F}$ on open of v.
- proper push forward

$$\frac{\varepsilon_v(Rf_*(\mathcal{F}\otimes\mathcal{G}))}{\varepsilon_v(Rf_*\mathcal{F})^{\mathrm{rk}\mathcal{G}}} = \begin{cases} \det(\mathcal{G}_X(\mathrm{ord}_v(\omega)\cdot cc(\mathcal{F}|X_v))) & \text{if } v\notin\Sigma,\ \mathrm{ord}_v(\omega)\neq 0 \\ \det\mathcal{G}(\mathcal{G}_X(\mathrm{dimtot}R\Phi_{\bar{u}_i}\cdot[u_i])) & \text{if } v\in\Sigma,\ v=f(u_i) \end{cases}$$

use $R\Gamma(X_{\bar{v}},\mathcal{F})\to R\Gamma(X_{\bar{\eta}},\mathcal{F})\to R\Phi_{\bar{u}_i}(\mathcal{F},f)\to$

Proof of Key Lemma

$$\begin{tikzcd} X \arrow[r, hook, "s_2"] \arrow[dr, "s_1"'] & T^*Y\times_Y X \arrow[d, "df"] \\ & T^*X \end{tikzcd}$$

$\underline{Z = CC\mathcal{F}}$ on T^*X.

$$X \xrightarrow{[\omega]} T^*Y\times_Y X$$

$$df^!(Z) = A + \sum_{i=1}^{m} b_i\cdot[T^*Y\times_Y u_i]$$

$$b_i = (df^!(Z), [\omega])_{T^*X,u_i} = (Z, f^*\omega)_{T^*X,u_i}$$
$$= -\mathrm{dimtot}R\Phi_{u_i}(\mathcal{F},f)$$

$$s_2^!A = c_1(f^*\Omega_Y^1)\cap A = \sum_{v\in|Y|}\mathrm{ord}_v(\omega)(i_v^*Z, T^*_{X_v}X_v)_{T^*X_v}$$
$$= -\sum_{v\in|Y|}\mathrm{ord}_v(\omega)(cc_{X_v}\mathcal{F}|X_v, T^*_{X_v}X_v)_{T^*X_v}.$$

$s_1^!A$
$\|$
$cc_X\mathcal{F} = (X, CC\mathcal{F}) = s_2^!df^!(CC\mathcal{F}) = \cdots$

§4 Swan classes

$$U \underset{\text{open}}{\overset{j}{\hookrightarrow}} X$$ ← proper smooth

$\mathcal{F}$: smooth étale sheaf on U.

In 2004, Kato and T. Saito define the Swan classes $\mathrm{Sw}^{\mathrm{ks}}(\mathcal{F})\in CH_0(X\setminus U)\otimes\mathbb{Q}$

The $Sw^{ks}(-)$ satisfies the following properties:

(1) If $\mathcal{F}$ and $\mathcal{G}$ have <u>same wild ramification</u> $\Rightarrow$ universally same Euler characteristic $\Rightarrow$ same rank, same artin conducts by cut-by-curve

then $Sw^{ks}\mathcal{F} = Sw^{ks}\mathcal{G}$

(2) <u>push-forward</u> For any Cartesian diagram

$$\begin{array}{ccc} V & \hookrightarrow & Y \\ \downarrow f & & \downarrow f \\ U & \hookrightarrow & X \end{array}$$

Y, X proper smooth
f: finite étale.
$\mathcal{G}$: smooth sheaf on V.

<u>then</u> $Sw^{ks}(f_*\mathcal{G}) = f_* Sw^{ks}(\mathcal{G}) + \operatorname{rank}\mathcal{G} \cdot Sw^{ks}(f_*\Lambda)$ in <u>$CH_0(X)$</u>.

(3) If $D = X\setminus U$ and $B = Y\setminus V$ are SNC divisor, then

$$Sw^{ks}(f_*\Lambda) = d^{\log} := (-1)^{\dim X-1} f_*\left\{\left(C_d(\Omega^1_{Y/k}(\log(B)) - f^*\Omega^1_{X/k}(\log D)\right) \cap [Y]\right\}$$

Hur Hurwitz logarithmic differential zero class

For $Sw^{cc}\mathcal{F} = \langle T^*_X X, \operatorname{rank}\mathcal{F}\cdot CC j_!\Lambda - CC j_!\mathcal{F}\rangle_{T^*X} \in CH_0(X)$.

$Sw^{cc}\mathcal{F} \in (3)$

$Sw^{cc}\mathcal{F} \in (1)$ Saito-Yatagawa, <u>a generalization</u> of Illusie and I. Vidal

<u>(2)</u> open.

<u>Proposition</u> Assume resolution of singularities in the strong sense.

Any $Sw^{\bullet}$ satisfying (1)(2)(3) is equal to Sw^{ks}.

<u>proof</u> Brauer induction $\Rightarrow$ WMA $\mathcal{F}$ is of rank 1, and trivialized by a finite étale cover of Galois group $\mathbb{Z}/p^n\mathbb{Z}$.

By induction on $n \Rightarrow$ <u>WMA $n=1$</u>

$\mathcal{F} \Longleftrightarrow \chi$: character χ of $G = \mathbb{F}_p$.
non-trivial

For any non-trivial character χ, and $\chi' \Rightarrow$ χ and χ' have same wild ramification.

$$\begin{array}{c} V \\ \downarrow f \\ U \end{array} \quad f_*\Lambda = \bigoplus_{\chi} \chi$$

constant $\mathcal{F}$; f of group $\mathbb{F}_p$.

$$(p-1)Sw^{\bullet}\mathcal{F} = \sum_{\chi\neq 1} Sw^{\bullet}\chi = Sw^{\bullet} f_*\Lambda = Sw^{ks} f_*\Lambda = (p-1) Sw^{ks}\mathcal{F}$$

Same wild ramification

X/k of finite type.

$\Lambda = \mathbb{F}_\ell$, $\ell \neq p = \mathrm{char}\, k$.

1) Assume X is normal and separated.

$\mathcal{F}$ and $\mathcal{F}'$ are locally constant constructible sheaves of Λ-modules.

We say $\mathcal{F}$ and $\mathcal{F}'$ have the same wild ramification if

$\exists$ proper normal $\bar{X} \underset{\substack{\text{dense}\\\text{open}}}{\supseteq} X$ such that for all geometric $\bar{x} \to \bar{X}$, we have

Let G be a finite quotient group of the inertia group $I_{\bar{x}} = \pi_1(\bar{X}_{(\bar{x})} \times_{\bar{X}} X, \bar{t})$ with respect to a base point $\bar{t}$ such that the pull-backs to $\bar{X}_{(\bar{x})} \times_{\bar{X}} X$ of $\mathcal{F}$ and $\mathcal{F}'$ correspond to G-modules M and M' respectively.

Then for every element $\sigma \in G$ of p-power order, we have an equality of the dimensions of the σ-fixed parts:

$$\dim M^\sigma = \dim M'^\sigma.$$

2) Let $\mathcal{F}$ and $\mathcal{F}'$ be constructible complexes of Λ-module on X.

We say $\mathcal{F}$ and $\mathcal{F}'$ have same wild ramification if

$\exists$ finite partition $X = \coprod_{i \in I} X_i$ by locally closed normal and separated subschemes such that for every q and for every i, the restrictions $\mathcal{H}^q(\mathcal{F})\big|_{X_i}$ and $\mathcal{H}^q(\mathcal{F}')\big|_{X_i}$ are locally constant constructible and have the same wild ramification in the sense of 1.