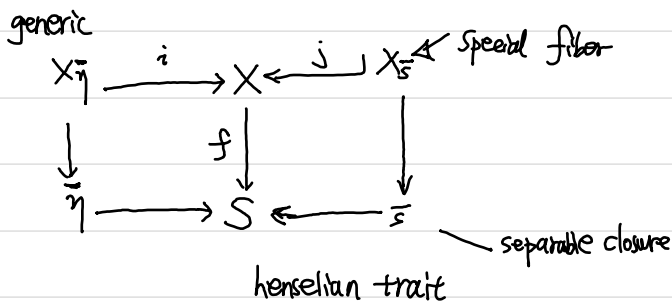


Cohomological Milnor formula and non-acyclicity classes for constructible étale sheaves.

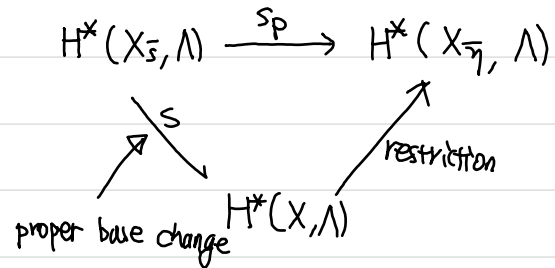
— Ramification theory from cohomological point of view.

Joint with Tigeng Zhao

1964.10.30, in a letter to Serre, Grothendieck first mentioned the "vanishing cycles".



We have the specialization when f is proper.



Question: When $H^*(X_{\bar{s}}, \Lambda) \longrightarrow H^*(X_{\bar{\eta}}, \Lambda)$ is an isomorphism?

This obstruction is controlled by the vanishing cycle groups.

Vanishing cycle

For $\mathcal{K} \in D_c^b(X, \Lambda)$, nearby cycle $R\Phi(\mathcal{K}, f) = i^* Rj_* j^* \mathcal{K}$

The vanishing cycle $R\Phi(\mathcal{K}, f) \in D_c^b(X_{\bar{s}}, \Lambda)$ sits in the distinguished triangle

$$\mathcal{K}|_{X_{\bar{s}}} \longrightarrow R\Phi(\mathcal{K}, f) \longrightarrow R\Phi(\mathcal{K}, f) \xrightarrow{+1}$$

when f is proper, it gives rise to a long exact sequence

$$H^i(X_{\bar{s}}, R\Phi(\mathcal{K})) \longrightarrow H^i(X_{\bar{s}}, \mathcal{K}) \xrightarrow{sp} H^i(X_{\bar{\eta}}, \mathcal{K}) \longrightarrow H^i(X_{\bar{s}}, R\Phi(\mathcal{K}))$$

If $R\Phi(\mathcal{K}) = 0 \Rightarrow sp$ is an isom.

In general, $R\Phi(\mathcal{K}) = 0 \iff f$ is locally acyclic (hence universally by Gabber) relatively to \mathcal{F} .

Lu-Zheng

We simply say: \mathcal{F} is ULA over S .

For general separated morphism $f: X \rightarrow S$, can also define ULA condition. We omit the details, but roughly say, its pullback to local trait is ULA.

Example If $X \xrightarrow{f} S$ is smooth $\Rightarrow \Lambda$ is ULA over S
 \parallel
 constant sheaf

If f has an isolated singularity at $x \in X$ $\Rightarrow \Lambda$ is not ULA at x .

In general, non-ULA points of $(\begin{smallmatrix} X \\ \downarrow \\ S \end{smallmatrix}, \mathcal{F})$ can be regarded as "singular points" associated to $(\begin{smallmatrix} X \\ \downarrow \\ S \end{smallmatrix}, \mathcal{F})$.

Singular locus = non-locally acyclicity locus = NA locus.

Invariants associated to ^{isolated} NA points

Conjecture on Milnor formula (1973, Deligne)

Bloch's conductor formula (1987, Bloch)

X : smooth of dim n

Y : smooth curve,

$y \in Y$

$X \xrightarrow[\text{flat}]{f} Y$ smooth over $Y \setminus \{y\}$.

total dimension

$$C_{n, X_y}^X(\Omega_{X/Y}^1) \cap [X] \in \text{CH}_0(X_y)$$

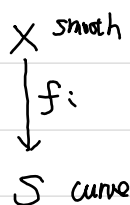
Milnor numbers

$$(-1)^n \text{length } \mathcal{O}_{X, x} \left\{ \text{Ext}_{\mathcal{O}_x}^i(\Omega_{X/S}^1, \mathcal{O}_x) \right\}_x$$

(1) If f has isolated singularity at $x \in X_y$, then $\dim \text{tot } R\Phi(\Lambda, f)_x = (-1)^n \deg(\text{localized chern class})$

(2) $a_y(Rf_* \Lambda) = \chi(X_{\bar{y}}) - \chi(X_y) + \text{Sw}(X_{\bar{y}}/\eta) \stackrel{\text{conductor formula}}{=} (-1)^n \deg(C_{n, X_y}^X(\Omega_{X/Y}^1) \cap [X])$
 $\text{Sw} \parallel H^*(X_{\bar{y}}, \Lambda) \hookrightarrow \mathcal{O}_{\text{all}}(\bar{y}/\eta)$

T. Saito : extends these two formula to ^{any} $\mathcal{F} \in \mathcal{D}_c^b(X, \Lambda)$ for smooth schemes by using characteristic cycle $CC\mathcal{F}$ of \mathcal{F} .



$x \in |X|$ isolated char point w.r.t $SS\mathcal{F}$ (the singular support)

$$-\dim \text{tot } R\mathbb{I}_x(\mathcal{F}) = \langle CC\mathcal{F}, df \rangle_{T^*X, x}$$

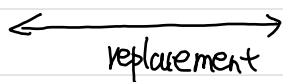
↑
(2017, T. Saito)

Today: We propose a cohomological way to ramification theory.

smooth variety

Singular variety

characteristic cycle



Relative cohomological char class
NA class

Notation

$$\Delta := \left(\begin{array}{ccc} Z \hookrightarrow X & \xrightarrow{f} & Y \\ & \searrow h & \swarrow g: \text{smooth} \\ & S & \end{array} \right), \quad \mathcal{K}_{X/S} = R^1 h^* \Lambda$$

$$\mathcal{F} \in \mathcal{D}_c^b(\Delta) \Leftrightarrow \mathcal{F} \in \mathcal{D}_c^b(X, \Lambda) \text{ s.t. } \mathcal{F} \text{ is } h\text{-ULA}$$

\mathcal{F} is f -ULA outside Z

$Z \sim$ NA locus of \mathcal{F}

an object $\mathcal{K}_\Delta \in \mathcal{D}_c^b(X, \Lambda)$, $C_\Delta(\mathcal{F}) \in H_Z^0(X, \mathcal{K}_\Delta)$

I will introduce a class $C_\Delta(\mathcal{F})$ supported on Z , which is compatible with proper push-forward and pull-backs.

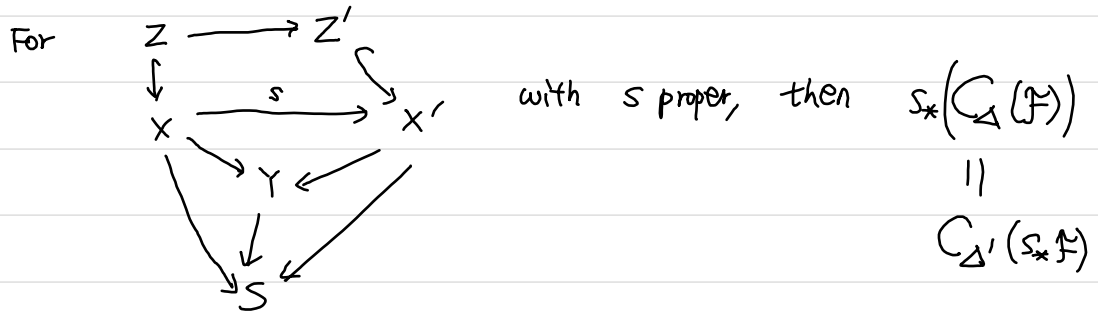
$$H^0(X, \mathcal{K}_X) = H^0(Z, \mathcal{K}_{Z/S})$$

When Z is small, i.e., $H^0(Z, \mathcal{K}_{Z/S}) = H^1(Z, \mathcal{K}_{Z/S}) = 0$, then $C_\Delta(\mathcal{F}) \in H^0(Z, \mathcal{K}_{Z/S})$

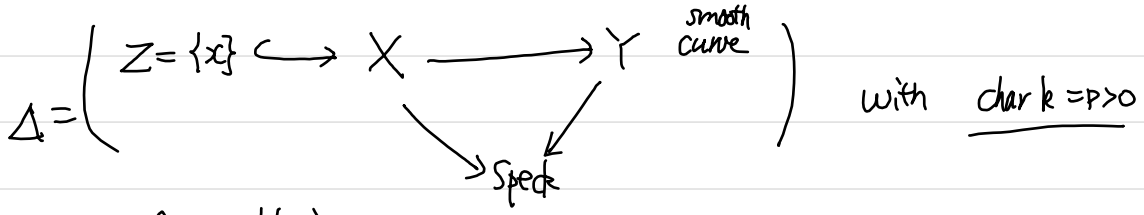
Thm (Y-Zhao) $\begin{pmatrix} Z' \hookrightarrow X' \rightarrow Y' \\ \downarrow \quad \downarrow \\ S' \end{pmatrix}$

(1) $S' \xrightarrow{b} S$, get Δ' by base change, then $L_X^* C_\Delta(\mathcal{F}) = C_{\Delta'}(\mathcal{F}')$

(2)



(3) Cohomological Milnor formula: Take

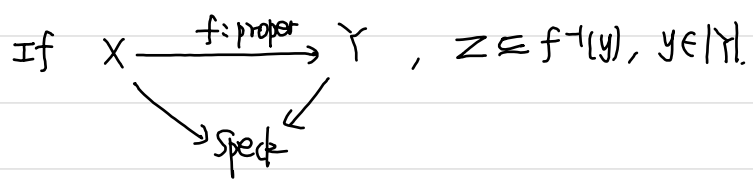


$$\mathcal{F} \in D_c^b(\Delta)$$

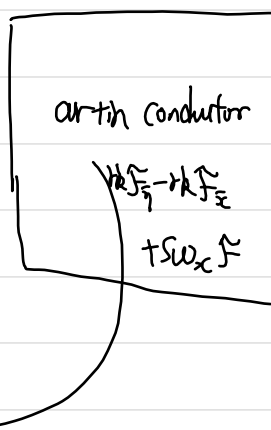
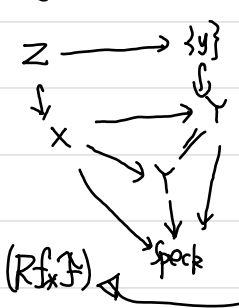
$$\text{Then } C_\Delta(\mathcal{F}) = -\dim \text{tot } R\Phi_x(\mathcal{F}, f)$$

$$H^0(x, \Lambda) = \Lambda$$

(4) Cohomological conductor formula: (with $\text{char } k = p > 0$)



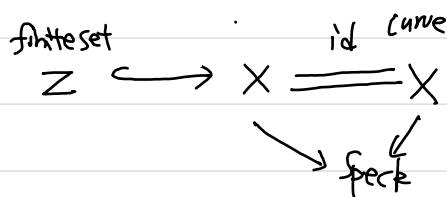
Apply (2) to



get $f_* C_\Delta(\mathcal{F}) = C_{Y/Y/k}(f_* \mathcal{F})$

Milnor $-\dim \text{tot } R\mathcal{O}_y(R\mathcal{F}_X, \text{id}) = -\alpha_y(R\mathcal{F}_X \mathcal{F})$

(5) cohomological Groth-Ogg-Shafarevich formula (with $\text{char } k = p > 0$)



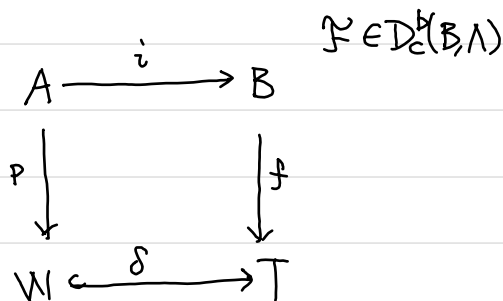
$\mathcal{F} \in D_c^b(\Lambda)$, i.e., \mathcal{F} smooth on $X \setminus Z$

Then
$$Q_\Delta(\mathcal{F}) = - \sum_{x \in Z} a_x(\mathcal{F}) \cdot [x] \quad \text{in } H^0(Z, \mathbb{Z}/k)$$

Construction of $Q_\Delta(\mathcal{F})$

Let me first introduce a def (transversal condition)

Consider a Cartesian diagram



We define a pull-back functor $\delta^! : D_c^b(B, \Lambda) \rightarrow D_c^b(A, \Lambda)$ such that

$$i^* \mathcal{F} \otimes p^* \mathcal{L} \xrightarrow{\quad} i^! \mathcal{F} \xrightarrow{\quad} \delta^! \mathcal{F} \xrightarrow{+1}$$

adj to
$$i_!(i^* \mathcal{F} \otimes p^* \mathcal{L}) = \mathcal{F} \otimes i_! p^* \mathcal{L} \cong \mathcal{F} \otimes f_* \delta_* \mathcal{L} \xrightarrow{\text{adj}} \mathcal{F}$$

when $\delta^! \mathcal{F} = 0 \iff \delta$ is \mathcal{F} -transversal. (This is related to ULA)

Given $\Delta = \begin{pmatrix} Z \xrightarrow{\quad} X \xrightarrow{f} Y \\ \quad \quad \quad \searrow h \quad \swarrow g \\ \quad \quad \quad S \end{pmatrix}$ and $\mathcal{F} \in D_c^b(\Delta, \Lambda)$.

Form the diagram

$$\begin{array}{ccc}
 X & \xlongequal{\quad} & X \\
 \downarrow \delta_1 & & \downarrow \delta_0 \\
 X \times_Y X & \xrightarrow{i} & X \times_S X \\
 \downarrow p & & \downarrow f \times f \\
 Y & \xrightarrow{\delta} & Y \times_S Y
 \end{array}$$

$\mathcal{F} \boxtimes_S D_{X/S}(\mathcal{F})$, $D_{X/S} \mathcal{F} = R\Gamma_{\text{an}}(\mathcal{F}, \mathcal{K}_{X/S})$

$$\mathcal{K}_{X/S} \otimes f^* \delta^! \Lambda$$

$$\begin{array}{ccccc}
 \mathcal{K}_{X/S} \otimes f^* \delta^! \Lambda & \xrightarrow{\text{is}} & \mathcal{K}_{X/S} & \longrightarrow & \delta^! \mathcal{K}_{X/S} \xrightarrow{+1} \\
 & & & & \text{!!} \\
 & & & & \mathcal{K}_\Delta
 \end{array}$$

$$\delta_1^* (i^* (\mathcal{F} \boxtimes_S D_{X/S}(\mathcal{F})) \otimes p^* \delta^! \Lambda) \longrightarrow \delta_1^* i^! (\mathcal{F} \boxtimes_S D_{X/S}(\mathcal{F})) \xrightarrow{\delta_1^*} \delta^! (\mathcal{F} \boxtimes_S D_{X/S}(\mathcal{F})) \xrightarrow{+1}$$

Technical Lemma $\delta_1^* \delta^! (\mathcal{F} \boxtimes_S D_{X/S}(\mathcal{F}))$ is supported on Z .

回顾更多一些.

$$\mathcal{C}_{X/S}(\mathcal{F}) = (\Lambda \longrightarrow \delta_0^! (\mathcal{F} \boxtimes_S D_{X/S}(\mathcal{F})) \longrightarrow \mathcal{K}_{X/S}) \text{ in } H^0(X, \mathcal{K}_{X/S})$$

↑ cohomological characteristic class.

apply $\delta_1^* (i^* (-) \otimes p^* \delta^! \Lambda) \longrightarrow \delta_1^* i^! (-) \longrightarrow \delta_1^* \delta^! (-) \xrightarrow{+1}$ to

$$\delta_0^! \Lambda = \delta_0^* \Lambda \longrightarrow \mathcal{F} \boxtimes_S^L D_{X/S}(\mathcal{F}) \longrightarrow \delta_0^* \mathcal{K}_{X/S}$$

Will get

$$\begin{array}{ccccc}
 f^* \delta^! \Lambda & & & & \mathcal{K}_{X/Y} \\
 \downarrow & & & & \downarrow \\
 \Lambda \longrightarrow & \delta_1^* i^! (\mathcal{F} \boxtimes_S D_{X/S}(\mathcal{F})) & \longrightarrow & \mathcal{K}_{X/S} & \longrightarrow \\
 \downarrow \delta_0^* \delta^! \Lambda & \downarrow & \searrow & \downarrow & \\
 & \longrightarrow & \delta_1^* \delta^! (\mathcal{F} \boxtimes_S D_{X/S}(\mathcal{F})) & \longrightarrow & \delta_1^* \delta^! \delta_0^* \mathcal{K}_{X/S} \\
 & & & & \text{is} \\
 & & & & \delta^! \mathcal{K}_{X/S}
 \end{array}$$

The composition $\mathcal{C} \rightarrow$ defines a class $\mathcal{C}_\Delta(\mathcal{F}) \in H_Z^0(X, \delta^! \mathcal{K}_{X/S})$ $\mathcal{K}_\Delta = \delta^! \mathcal{K}_{X/S}$.

We call K_Δ the non-acyclicity class of f .

For $\Delta = \left(\begin{array}{ccc} Z \hookrightarrow X & \xrightarrow{f} & Y \\ & \searrow \scriptstyle g \text{ smooth} & \end{array} \right)$. Assume $H^*(Z, K_{Z/S}) = H^1(Z, K_{Z/S}) = 0$.

We expect the following formula holds:

$$C_{X/S}(\mathcal{F}) = \underbrace{C_r(f^* \Omega_{Y/S}^{1, V})}_{r = \text{rel. dim of } g} \cap C_{X/Y}(\mathcal{F}) + C_\Delta(\mathcal{F}) \quad \text{in } H^0(X, K_{X/S})$$

- (r-Zhao) If $Z = \emptyset$
- (Abbes-Saito) If $f = \text{id}$ and $S = \text{Spec } k$
- If $S = \text{Spec } k$ and Y is a smooth curve, and if $Z =$ finite sets of closed points,

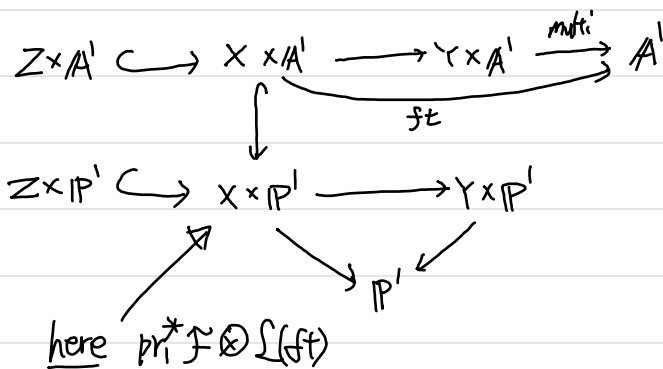
then

$$C_{X/k}(\mathcal{F}) = \underbrace{C_r(f^* \Omega_{Y/k}^{1, V})}_{\uparrow \text{ is "1" }} \cap C_{X/Y}(\mathcal{F}) + C_\Delta(\mathcal{F})$$

$$= \sum_{x \in Z} \dim \text{Tot } R\Phi_x(\mathcal{F}, f) \cdot [x].$$

Idea of the proof (May assume $Y = \mathbb{A}^1$)

$\mathcal{L} =$ Artin-Schreier sheaf on \mathbb{A}^1 associated with ψ



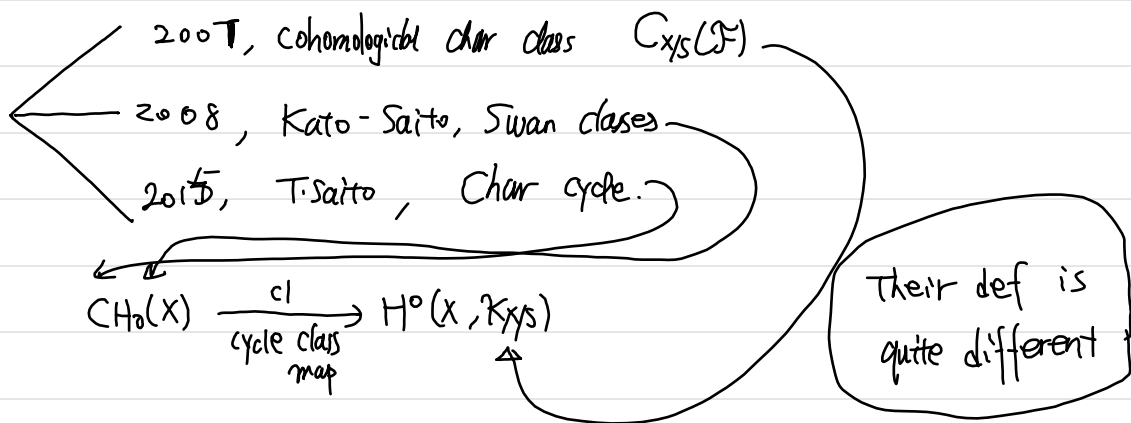
After taking a finite $\sqrt{\text{surj}}$ extension $\mathbb{P} \rightarrow \mathbb{P}'$, we may assume

$$\text{pr}_1^* \mathcal{F} \otimes \mathcal{L}(ff) \in D_c^b(\Delta_{\mathbb{P}}) \text{ with } \Delta_{\mathbb{P}} = \left(\begin{array}{ccc} Z \times \mathbb{P} & \hookrightarrow & X \times \mathbb{P} \longrightarrow Y \times \mathbb{P} \\ & & \downarrow \mathbb{P} \swarrow \end{array} \right)$$

Apply pull-back property:

$$\begin{array}{c}
 C_{\Delta \times \mathbb{P}^1}(\mathcal{F}) \in H^0(Z, \mathcal{K}_{Z/\mathbb{P}^1}) = \bigoplus_{x \in Z} \mathbb{1} \\
 \begin{array}{ccc}
 & \swarrow \text{"\sigma"} & \searrow \text{"\sigma"} \\
 & & \\
 \end{array} \\
 C_{\Delta}(\underbrace{\mathbb{P}_{\mathbb{P}^1}(\mathcal{F}^* \otimes \mathcal{O}_Z(-1))}_{\text{supported on } Z}) \equiv C_{\Delta}(\mathcal{F}) \\
 \parallel \\
 - \sum_{x \in Z} \dim \text{tot } R\Phi_x(\mathcal{F}) \cdot [x]
 \end{array}$$

Now we apply our thm to confirm proj case of Saito's conjecture



They are the same in $H^0(X, K_{X/S})$.

[Thm] True for projective smooth (if assume resolution of singularity,)
 true for quasi-proj

proof. prove that they all satisfy a Milnor formula and a filtration formula, then prove by induction. ▣