Cohomological Milnor formula and non-acyclicity Classes for constructible étale sheaves.
— Ramification theory from cohomological point of view.

Joint with Tigeng Zhao

1964.10.30, in a letter to Serre, Grothendieck first mentioned the "vanishing cycles".

generic

$$\begin{tikzcd} X_{\bar\eta} \arrow[r, "i"] \arrow[d] & X \arrow[d, "f"] & X_{\bar s} \arrow[l, hook', "j"'] \arrow[d] \\ \bar\eta \arrow[r] & S & \bar s \arrow[l] \end{tikzcd}$$

Special fiber

separable closure

henselian trait

We have the specialization when f is proper.

$$\begin{tikzcd} H^*(X_{\bar s},\Lambda) \arrow[rr, "sp"] & & H^*(X_{\bar\eta},\Lambda) \\ & H^*(X,\Lambda) \arrow[ul, "s"] \arrow[ur, "\text{restriction}"'] & \end{tikzcd}$$

proper base change

Question: When $H^*(X_s,\Lambda) \longrightarrow H^*(X_{\bar\eta},\Lambda)$ is an isomorphism?

This obstruction is controlled by the vanishing cycle groups.

<u>Vanishing cycle</u> For $\mathcal{K} \in D_c^b(X,\Lambda)$, nearby cycle $R\Psi(\mathcal{K},f) = i^*Rj_*j^*\mathcal{K}$

The vanishing cycle $R\Phi(\mathcal{K},f) \in D_c^b(X_{\bar s},\Lambda)$ sits in the distinguished triangle

$$\mathcal{K}|_{X_{\bar s}} \longrightarrow R\Psi(\mathcal{K},f) \longrightarrow R\Phi(\mathcal{K},f) \xrightarrow{+1}$$

when f is proper, it gives rise to a long exact sequence

$$H^{i-1}(X_{\bar s}, R\Phi(K)) \longrightarrow H^i(X_{\bar s},K) \xrightarrow{sp} H^i(X_{\bar\eta},K) \to H^i(X_{\bar s},R\Phi(K))$$

If $R\Phi(K)=0 \Rightarrow sp$ is an isom.

In general, $R\Phi(K)=0 \longleftrightarrow f$ is locally acyclic (hence universally by Gabber $\overset{\text{Lu-Zheng}}{)}$ relatively to $\mathcal{F}$.

We simply say: $\mathcal{F}$ is ULA. over S.

For general separated morphism $f: X \longrightarrow S$, can also define ULA. condition. We omit the details, but roughly say, its pull-back to local trait is ULA.

Example If $X \xrightarrow{f} S$ is smooth $\Rightarrow$ $\underset{\text{constant sheaf}}{\overset{\parallel}{\Lambda}}$ is ULA over S

If f has an isolated singularity at $x \in |X|$ $\Rightarrow$ Λ is not ULA at x.

In general, non-ULA points of $\left(\begin{smallmatrix} X \\ \downarrow \\ S \end{smallmatrix}, \mathcal{F}\right)$ can be regarded as "singular points" associated to $\left(\begin{smallmatrix} X \\ \downarrow \\ S \end{smallmatrix} \mathcal{F}\right)$.

Singular locus $=$ non-locally acyclicity locus. $=$ NA locus.

Invariants associated to isolated NA points

Conjecture on Milnor formula (1973, Deligne)

Bloch's conductor formula (1987, Bloch)

X: smooth of dim n

Y: smooth curve,

$y \in |Y|$

$X \xrightarrow[\text{flat}]{f} Y$ smooth over $Y \setminus \{y\}$.

Milnor numbers

$(-1)^n \operatorname{length}_{\mathcal{O}_{X,x}} \{ \mathcal{E}xt^1_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{O}_X) \}_x$

total dimension

$C^X_{n\, X_y}(\Omega^1_{X/Y}) \cap [X] \in CH_0(X_y)$

(1) If f has isolated singularity at $x \in X_y$, then $\operatorname{dimtot} R\Phi(\Lambda, f)_x \overset{(-1)^n}{=} \deg\left(\text{localized chern class}\right)$

(2) $a_y(Rf_*\Lambda) := \chi(X_{\bar\eta}) - \chi(X_{\bar y}) + \underset{\overset{\parallel}{Sw\, H^*(X_{\bar\eta}, \Lambda)} \circlearrowleft Gal(\bar\eta/\eta)}{Sw(X_\eta/\eta)} \underset{\text{conductor formula}}{=} (-1)^n \deg\left(C^X_{n\, X_y}(\Omega^1_{X/Y}) \cap [X]\right)$

T. Saito : extends these two formula to any $\mathcal{F} \in D_c^b(X, \Lambda)$ for smooth schemes by using characteristic cycle $CC\mathcal{F}$ of $\mathcal{F}$.

$$\begin{array}{l} X^{\text{smooth}} \\ \big\downarrow f: \\ S \ \text{curve} \end{array}$$

$x \in |X|$ isolated char point w.r.t $SS\mathcal{F}$ (the singular support)

$$-\dim\mathrm{tot}\, R\Phi_x(\mathcal{F}) = \langle CC\mathcal{F}, df \rangle_{T^*X, x}$$

(2017, T. Saito)

<u>Today</u>: We propose a cohomological way to Ramification theory.

smooth variety		Singular variety
characteristic cycle	$\longleftrightarrow$ replacement	$<$ Relative cohomological char class, NA class

<u>Notation</u>

$$\Delta := \left(\begin{tikzcd} Z \arrow[r, hook] & X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g: \text{smooth}"] \\ & & S \end{tikzcd} \right), \quad K_{X/S} = Rh^! \Lambda$$

$\mathcal{F} \in D_c^b(\Delta) \Leftrightarrow \mathcal{F} \in D_c^b(X, \Lambda)$ s.t $\mathcal{F}$ is h-ULA

$\mathcal{F}$ is f-ULA outside Z

$Z \sim$ NA locus of $\mathcal{F}$

I will introduce $\begin{cases} \text{an object } K_\Delta \in D_c^b(X, \Lambda), \ C_\Delta(\mathcal{F}) \in H_Z^0(X, K_\Delta) \\ \text{a class } C_\Delta(\mathcal{F}) \text{ supported on } Z, \text{ which is compatible with} \end{cases}$ proper push-forward and pull-backs.

$$H^0_Z(X, K_\Delta) = H^0(Z, K_{Z/S})$$

When Z is small, i.e., $H^0(Z, K_{Z/Y}) = H^1(Z, K_{Z/Y}) = 0$, then $C_\Delta(\mathcal{F}) \in H^0(Z, K_{Z/S})$

<u>Thm (Y-Zhao)</u>

$$\Delta' = \left(\begin{tikzcd} Z' \arrow[r, hook] & X' \arrow[r] \arrow[dr] & Y' \arrow[d] \\ & & S' \end{tikzcd}\right)$$

(1) $S' \xrightarrow{b} S$, get Δ' by base change, then $b_X^* C_\Delta(\mathcal{F}) = C_{\Delta'}(\mathcal{F}')$

(2)

For
$$\begin{tikzcd} Z \arrow[r] \arrow[d, hook] & Z' \arrow[d, hook] \\ X \arrow[r, "s"] \arrow[dr] \arrow[ddr] & X' \arrow[d] \arrow[dd] \\ & Y \arrow[d] \\ & S \end{tikzcd}$$
with s proper, then
$$s_*(C_\Delta(\mathcal{F})) = C_{\Delta'}(s_*\mathcal{F})$$

(3) Cohomological Milnor formula: Take

$$\Delta = \left(\begin{tikzcd} Z = \{x\} \arrow[r, hook] & X \arrow[r] \arrow[dr] & Y \arrow[d] & \text{smooth curve} \\ & & \operatorname{Spec} k \end{tikzcd}\right) \quad \text{with } \underline{\operatorname{char} k = p > 0}$$

$\mathcal{F} \in D^b_c(\Delta)$

Then $C_\Delta(\mathcal{F}) = -\operatorname{dimtot} R\Phi_{\bar{x}}(\mathcal{F}, f)$

$\in$

$H^0(x, \Lambda) = \Lambda$

(4) Cohomological conductor formula: (with $\operatorname{char} k = p > 0$)

If $\begin{tikzcd} X \arrow[r, "f:\text{proper}"] \arrow[dr] & Y \arrow[d] \\ & \operatorname{Spec} k \end{tikzcd}$, $Z \subseteq f^{-1}(y)$, $y \in |Y|$. Apply (2) to

$$\begin{tikzcd} Z \arrow[r] \arrow[d, hook] & \{y\} \arrow[d, hook] \\ X \arrow[r] \arrow[dr] \arrow[ddr] & Y \arrow[d, equal] \arrow[dd] \\ & Y \arrow[d] \\ & \operatorname{Spec} k \end{tikzcd}$$

artin conductor

$\operatorname{rk} \mathcal{F}_{\bar\eta} - \operatorname{rk} \mathcal{F}_{\bar x} + \operatorname{Sw}_x \mathcal{F}$

get $f_* C_\Delta(\mathcal{F}) = C_{Y/Y/k}(f_*\mathcal{F})$

$\underset{\text{Milnor}}{=} -\operatorname{dimtot} R\Phi_y(Rf_*\mathcal{F}, \mathrm{id}) = -a_y(Rf_*\mathcal{F})$

(5) cohomological Groth-Ogg-Shafarevich formula (with char $k = p > 0$)

$$\begin{tikzcd} \underset{\text{finite set}}{Z} \arrow[r, hook] & X \arrow[r, equal, "\text{id}"] \arrow[dr] & \overset{\text{curve}}{X} \arrow[d] \\ & & \operatorname{Spec} k \end{tikzcd}$$

$\mathcal{F} \in D_c^b(\Lambda)$, i.e, $\mathcal{F}$ smooth on $X \setminus Z$

Then $C_\Delta(\mathcal{F}) = -\sum_{x \in Z} a_x(\mathcal{F}) \cdot [x]$ in $H^0(Z, K_{Z/k})$

Construction of $C_\Delta(\mathcal{F})$

Let me first introduce a def (transversal condition)

Consider a Cartesian diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "p"'] & B \arrow[d, "f"] \\ W \arrow[r, hook, "\delta"] & T \end{tikzcd}$$

$\mathcal{F} \in D_c^b(B, \Lambda)$

We define a pull-back funtor $\delta^\Delta : D_c^b(B, \Lambda) \longrightarrow D_c^b(A, \Lambda)$ such that

$$i^*\mathcal{F} \otimes p^*\delta^!\Lambda \longrightarrow i^!\mathcal{F} \longrightarrow \delta^\Delta\mathcal{F} \xrightarrow{+1}$$

adj to $i_!(i^*\mathcal{F} \otimes p^*\delta^!\Lambda) = \mathcal{F} \otimes i_!p^*\delta^!\Lambda \cong \mathcal{F} \otimes f^*\delta_!\delta^!\Lambda \xrightarrow{\text{adj}} \mathcal{F}$

when $\delta^\Delta\mathcal{F} = 0 \iff \delta$ is $\mathcal{F}$-transversal. (This is related to ULA)

Given $\Delta = \left(\begin{tikzcd} Z \arrow[r, hook] & X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & & S \end{tikzcd} \right)$ and $\mathcal{F} \in D_c^b(\Delta, \Lambda)$.

Form the diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "\delta_1"] \arrow[dd, bend right=60, "f"'] & X \arrow[d, "\delta_0"] \\ X\times_Y X \arrow[r, "i"] \arrow[d, "p"] & X\times_S X \arrow[d, "f\times f"] \\ Y \arrow[r, "\delta"'] & Y\times_S Y \end{tikzcd}$$

$\mathcal{K}_{X/S}$

$\mathcal{F}\boxtimes_S D_{X/S}(\mathcal{F})$. $\quad , D_{X/S}\mathcal{F} = R\mathcal{H}om(\mathcal{F}, \mathcal{K}_{X/S})$

$$\begin{tikzcd} \mathcal{K}_{X/S}\otimes f^*\delta^!\Lambda \arrow[d, phantom, "\wr\!\shortparallel"] & & & \\ \mathcal{K}_{X/Y} \arrow[r] & \mathcal{K}_{X/S} \arrow[r] & \delta^\Delta \mathcal{K}_{X/S} \arrow[r, "+1"] \arrow[d, phantom, "\shortparallel"] & {} \\ & & \mathcal{K}_\Delta & \end{tikzcd}$$

$$\delta_1^*\left(i^*(\mathcal{F}\boxtimes_S D_{X/S}(\mathcal{F}))\otimes p^*\delta^!\Lambda\right) \longrightarrow \delta_1^* i^!(\mathcal{F}\boxtimes_S D_{X/S}(\mathcal{F})) \longrightarrow \delta_1^*\delta^\Delta(\mathcal{F}\boxtimes_S D_{X/S}\mathcal{F}) \xrightarrow{+1}$$

<u>Technical</u> <u>Lemma</u> $\delta_1^*\delta^\Delta(\mathcal{F}\boxtimes_S D_{X/S}(\mathcal{F}))$ is supported on Z.

回顾更多一些.

$$C_{X/S}(\mathcal{F}) = \left(\Lambda \longrightarrow \delta_0^!(\mathcal{F}\boxtimes^L_S D_{X/S}(\mathcal{F})) \longrightarrow \mathcal{K}_{X/S}\right) \text{ in } H^0(X, \mathcal{K}_{X/S})$$

↑ cohomological characteristic class.

apply $\delta_1^*(i^*(-)\otimes p^*\delta^!\Lambda) \xrightarrow{\delta_1^*} i^!(\ ) \longrightarrow \delta_1^*\delta^\Delta(\ ) \xrightarrow{+1}$ to

$$\delta_{0!}\Lambda = \delta_{0*}\Lambda \longrightarrow \mathcal{F}\boxtimes^L_S D_{X/S}(\mathcal{F}) \longrightarrow \delta_{0*}\mathcal{K}_{X/S}$$

Will get

$$\begin{tikzcd} f^*\delta^!\Lambda \arrow[d] & & \mathcal{K}_{X/Y} \arrow[d] & \\ \Lambda \arrow[r] \arrow[d] \arrow[drr, bend left] & \delta_1^* i^!(\mathcal{F}\boxtimes^L_S D_{X/S}\mathcal{F}) \arrow[r] \arrow[d] & \mathcal{K}_{X/S} \arrow[r] \arrow[d] & {} \\ \delta_1^*\delta^\Delta\delta_{0*}\Lambda \arrow[r] & \delta_1^*\delta^\Delta(\mathcal{F}\boxtimes^L_S D_{X/S}\mathcal{F}) \arrow[r] & \delta_1^*\delta^\Delta\delta_{0*}\mathcal{K}_{X/S} \arrow[d, phantom, "\wr\!\shortparallel"] & \\ & & \delta^\Delta\mathcal{K}_{X/S} & \end{tikzcd}$$

The composition $\rightarrow$ defines a class $C_\Delta(\mathcal{F}) \in H^0_Z(X, \delta^\Delta\mathcal{K}_{X/S})$ $\quad \mathcal{K}_\Delta := \delta^\Delta\mathcal{K}_{X/S}$.

We call $\mathcal{K}_\Delta$ the non-acyclicity class of $\mathcal{F}$.

For $\Delta = \left(\begin{tikzcd} Z \arrow[r, hook] & X \arrow[r, "f"] \arrow[dr] & Y \arrow[d, "g \text{ smooth}"] \\ & & S \end{tikzcd}\right)$. Assume $H^0(Z, \mathcal{K}_{Z/S}) = H^1(Z, \mathcal{K}_{Z/S}) = 0$.

We expect the following formula holds:

$$C_{X/S}(\mathcal{F}) = \underset{\substack{\| \\ r = \text{rel. dim of } g}}{c_r}(f^* \Omega^{1,\vee}_{Y/S}) \cap C_{X/Y}(\mathcal{F}) + C_\Delta(\mathcal{F}) \quad \text{in } H^0(X, \mathcal{K}_{X/S})$$

- (r-Zhao) If $Z = \phi$
- (Abbes-Saito) If $f = \mathrm{id}$ and $S = \operatorname{Spec} k$
- If $S = \operatorname{Spec} k$ and Y is a smooth curve, and if Z = finite sets of closed points,

then
$$C_{X/k}(\mathcal{F}) = c_r(f^* \Omega^{1,\vee}_{Y/k}) \cap C_{X/Y}(\mathcal{F}) + C_\Delta(\mathcal{F})$$
$\uparrow$ 应是“1”
$$C_\Delta(\mathcal{F}) = -\sum_{x \in Z} \operatorname{dimtot} R\Phi_x(\mathcal{F}, f) \cdot [x].$$

<u>Ideal of the proof</u> (May assume $Y = \mathbb{A}^1$)

$\mathcal{L}$: Artin-Schreier sheaf on $\mathbb{A}^1$ associated with ψ

$$\begin{tikzcd} Z \times \mathbb{A}^1 \arrow[r, hook] & X \times \mathbb{A}^1 \arrow[r] \arrow[d, hook] \arrow[rr, bend right, "ft"'] & Y \times \mathbb{A}^1 \arrow[r, "\text{multi}"] & \mathbb{A}^1 \\ Z \times \mathbb{P}^1 \arrow[r, hook] & X \times \mathbb{P}^1 \arrow[r] \arrow[dr] & Y \times \mathbb{P}^1 \arrow[d] & \\ & & \mathbb{P}^1 & \end{tikzcd}$$

here $pr_1^* \mathcal{F} \otimes \mathcal{L}(ft)$

After taking a finite $\overset{\text{surj}}{\vee}$ extension $\mathbb{P} \longrightarrow \mathbb{P}^1$, we may assume

$pr_1^* \mathcal{F} \otimes \mathcal{L}(ft) \in D^b_c(\Delta \times \mathbb{P})$ with $\Delta \times \mathbb{P} = \left(\begin{tikzcd} Z \times \mathbb{P} \arrow[r, hook] & X \times \mathbb{P} \arrow[r] \arrow[dr] & Y \times \mathbb{P} \arrow[d] \\ & & \mathbb{P} \end{tikzcd}\right)$

Apply pull-back property:

$$\begin{tikzcd}
 & C_{\Delta\times\mathbb{P}}(\mathcal{F}) \in H^0(Z, \mathcal{K}_{Z/\mathbb{P}}) = \bigoplus_{x\in Z}\Lambda \arrow[dl, "``\infty"'] \arrow[dr, "``0"] & \\
\underbrace{C_\Delta(\Psi_{pr_2}(pr_1^*\mathcal{F}\otimes\mathcal{L}(ft)))}_{\text{supported on } Z} \arrow[rr, equal] \arrow[d, equal] & & C_\Delta(\mathcal{F}) \\
-\sum_{x\in Z} \operatorname{dimtot} R\Phi_x(\mathcal{F}, f)\cdot[x] & &
\end{tikzcd}$$

Now we apply our thm to confirm proj case of Saito's conjective

2007, cohomological char class $C_{X/S}(\mathcal{F})$

2008, Kato-Saito, Swan classes

2015, T.Saito, Char cycle.

$$CH_0(X) \xrightarrow[\text{cycle class map}]{cl} H^0(X, \mathcal{K}_{X/S})$$

Their def is quite different!

They are the same in $H^0(X, \mathcal{K}_{X/S})$.

$\boxed{\text{Thm}}$ True for projective smooth (if assume resolution of singularity, true for quasi-proj)

proof. prove that they all satisfy a Milnor formula and a fibration formula, then prove by induction. ∎