§5 Integral dependence

2024.04.10

Integral 定义 / Going-up / Going down (之前处理过 $\overset{\text{open map}}{\swarrow}$ flat map 的 going down) / 整闭包

Definition 5.1 $A \subseteq B$ subring with $1 \in A$.

$x \in B$ is integral over A iff x is a root of a monic polynomial with coeff in A:

$$\exists\, x^n + a_1 x^{n-1} + \dots + a_n = 0,\ a_i \in A.$$

$C = \{ b \in B \mid b \text{ is integral over } A \} \subseteq B$

$A \subseteq C \subseteq B$

— We will show that C is a subring, and we call C is the integral closure of A in B

比如：$\sqrt[5]{3} + \sqrt[5]{5}$ 能被首一整系数多项式零化.

— If $C = A$, then we say A is integrally closed in B.

— If $C = B$, then we say B is integral over A.

Example 5.2 The integral closure of $\mathbb{Z}$ in $\mathbb{Q}$ is $\mathbb{Z}$.

If $0 \neq x = \frac{r}{s} \in \mathbb{Q}$, $(r,s) = 1$, and if $\frac{r}{s}$ is integral over $\mathbb{Z}$,

then $\exists \left(\frac{r}{s}\right)^n + a_1 \left(\frac{r}{s}\right)^{n-1} + \dots + a_n = 0,\ a_i \in \mathbb{Z},\ a_n \neq 0$

$\Leftrightarrow r^n + a_1 r^{n-1} s + \dots + a_n s^n = 0 \Rightarrow s \mid r^n \underset{(s,r)=1}{\Rightarrow} s = \pm 1, \Rightarrow x = \frac{r}{s} \in \mathbb{Z}.$

Prop 5.3 TFAE:

(1) $x \in B$ is integral over A.

(2) $A[x]$ is f.g as A-module (finite A-algebra)

(3) $A[x]$ is contained in a subring $C \subseteq B$ such that C is f.g as A-module

(4) $\exists$ faithful $A[x]$-module M which is f.g as an A-module.

(若 $y \in A[x]$, $y \cdot M = 0$, then $y = 0$)

proof (1) ⇒ (2) If $x^n+a_1x^{n-1}+\dots+a_n=0$, then $A[x]$ is generated by $1, x, \dots, x^{n-1}$ as A-module.

(2) ⇒ (3) Take $C=A[x]$

(3) ⇒ (4) Take $M=C$, which is faithful $A[x]$-module

(若 $y\cdot C=0 \Rightarrow y\cdot 1=y=0$)

(4) ⇒ (1) Consider $M\xrightarrow{x}M$, M: f.g.

By corollary 1.4 $\Rightarrow \exists\, x^n+a_1x^{n-1}+\dots+a_n=0$ in $End(M)$.

Since M is faithful $\Rightarrow x^n+a_1x^{n-1}+\dots+a_n=0$ in $A[x]$.

Corollary 5.4 If $x_1,\dots,x_n\in B$ are integral over A, then $A[x_1,\dots,x_n]$ is a f.g. A-module. Thus $x_1\pm x_2$, $x_1\cdot x_2$ are integral over A.

⇒ the integral closure C of A in B is a ring.

Definition 5.5 $A\xrightarrow{f}B$ ring homo. If B is integral over $f(A)$, then we say f is integral, or that B is an integral A-algebra. 注意：并不要求 $A\to B$ injective.

Corollary 5.6 If f is integral and of finite type, then f is finite (as A-module) (as A-algebra)

Corollary 5.7 (transitivity of integral dependence)

$$\begin{tikzcd} C \\ B \arrow[u, "\text{integral}"'] \\ A \arrow[u, "\text{integral}"'] \end{tikzcd}$$

$A\subseteq B\subseteq C$ rings.

B integral over A, C integral over B.

Then C is integral over A.

proof Let $x\in C$. Then $\exists\, x^n+b_1x^{n-1}+\dots+b_n=0$ $(b_i\in B)$

The ring $B'=A[b_1,\dots,b_n]$ is f.g. A-module.

$\Rightarrow B'[x]$ is a f.g B-module

$\Rightarrow B'[x]$ is a f.g. A-module $\Rightarrow x$ is integral over A.

Corollary 5.8 $A \subseteq B$, $C =$ integral closure of A in B.

Then C is integrally closed in B.

proof If $x \in B$ integral over C, then x is integral over $A \Rightarrow x \in C$. ▦

Proposition 5.9 (Quotient & localization)

$A \subseteq B$ and B integral over A.

(1) For any $J \subseteq B$ ideal, $I := J^c = J \cap A$. Then B/J is integral over A/I.

pf $x \in B$, $\exists\, x^n + a_1 x^{n-1} + \cdots + a_n = 0$ with $a_i \in A$. Then apply mod J. ▦

(2) If $S \subseteq \cancel{B}\, A$ is multi closed, then $S^{-1}B$ is integral over $S^{-1}A$.

pf $\frac{x}{s} \in S^{-1}B$ $(s \in S)$. The above equation gives

$$\left(\frac{x}{s}\right)^n + \left(\frac{a_1}{s}\right)\cdot\left(\frac{x}{s}\right)^{n-1} + \cdots + \frac{a_n}{s^n} = 0 \Rightarrow \frac{x}{s} \text{ integral over } S^{-1}A.$$

现处理 going-up (之前证了 flat morphism 的 going-up), 后处理 going-down.

先证几个辅助结论.

Prop 5.10 $A \subseteq B$, B integral over A.

(1) If A, B are integral domain, then B is a field iff A is a field.

(2) $\mathfrak{q} \subseteq B$ prime, $\mathfrak{p} = \mathfrak{q} \cap A$. Then $\mathfrak{q}$ is maximal iff $\mathfrak{p}$ is maximal.

(pf : apply (1) to the integral map $A/\mathfrak{p} \longrightarrow B/\mathfrak{q}$).

(3) $\mathfrak{q} \subseteq \mathfrak{q}'$ prime ideals of B. If $\mathfrak{p} := \mathfrak{q}^c = \mathfrak{q}'^c$, then $\mathfrak{q} = \mathfrak{q}'$.

(下降的链保持等)

ⓟⓕ

$$\begin{array}{cccc} B & \mathfrak{q} \subseteq \mathfrak{q}' & B_{\mathfrak{p}} & \mathfrak{q}B_{\mathfrak{p}} \subseteq \mathfrak{q}'B_{\mathfrak{p}} \\ | & \wr & | & \quad \| \qquad\quad \| \\ A & \mathfrak{p} & A_{\mathfrak{p}} & \mathfrak{n} \subseteq \mathfrak{n}' \\ & & & \mathfrak{m} = \mathfrak{p}A_{\mathfrak{p}} \end{array}$$

$B_{\mathfrak{p}}$ is integral over $A_{\mathfrak{p}}$.

By (2) $\Rightarrow$ $\mathfrak{n} \subseteq \mathfrak{n}'$ are maximal $\Rightarrow \mathfrak{n} = \mathfrak{n}'$

$\Rightarrow \mathfrak{q} = \mathfrak{q}'$ by the correspondence $\left\{\begin{array}{c}\text{prime ideals} \\ \text{in } B_{\mathfrak{p}}\end{array}\right\} \xleftrightarrow{1:1} \left\{\begin{array}{c}\text{prime ideals in} \\ B \text{ which does not} \\ \text{meet } \mathfrak{p}B\end{array}\right\}$

there is no inclusion relation between the prime ideals of B lying over a fixed prime ideal of A.

<u>proof of (1)</u> $\Leftarrow$ If A is a field, let $0\neq y\in B$ with $y^n+a_1y^{n-1}+\cdots+a_n=0$ with $a_n\neq 0$.

$\Rightarrow y^{-1}=-a_n^{-1}(y^{n-1}+a_1y^{n-2}+\cdots+a_{n-1})\in B$

$\Rightarrow$ B is a field.

$\Rightarrow$ Suppose B is a field, Let $0\neq x\in A$.

Then $x^{-1}\in B \Rightarrow x^{-1}$ is integral over A.

$\Rightarrow \exists\ (x^{-1})^m+a_1'(x^{-1})^{m-1}+\cdots+a_m'=0 \quad (a_i'\in A)$.

$\Rightarrow x^{-1}=-(a_1'+a_2'x+\cdots+a_m'x^{m-1})\in A$

$\Rightarrow$ A is a field. ▦

<u>Thm 5.11</u> $A\subseteq B$ rings. B integral over A. Then $\mathrm{Spec}B\longrightarrow\mathrm{Spec}A$ is surjective.

(By 5.10 可证明：If A is local with maximal ideal $\mathfrak{p}$, then the prime ideals of B lying over $\mathfrak{p}$ are precisely the maximal ideals of B.)

<u>proof</u>

$$\begin{tikzcd}[column sep=small, row sep=small] & B_{\mathfrak{p}} \arrow[dd, dash] \\ B \arrow[ur, dash] \arrow[dd, dash] & \\ & A_{\mathfrak{p}} \\ A \arrow[ur, dash] & \end{tikzcd}$$

For any prime ideal $\mathfrak{p}\subseteq A$, we show that: $\exists$ prime ideal $\mathfrak{q}\subseteq$ such that $\mathfrak{q}\cap A=\mathfrak{p}$.

$B_{\mathfrak{p}}$ is integral over $A_{\mathfrak{p}}$. (<u>这里 $B_{\mathfrak{p}}=B\otimes_A A_{\mathfrak{p}}$</u>)

Let $\mathfrak{m}\subseteq B_{\mathfrak{p}}$ be the maximal ideal of $B_{\mathfrak{p}}$.

By 5.10 $\Rightarrow \mathfrak{m}\cap A_{\mathfrak{p}}$ is maximal $\Rightarrow \mathfrak{m}\cap A_{\mathfrak{p}}=\mathfrak{p}A_{\mathfrak{p}}$.

Let $\mathfrak{q}=B\cap\mathfrak{m}$, then $\mathfrak{q}$ is a prime of B and we have $\mathfrak{q}\cap A=\mathfrak{p}$.

<u>Thm 5.12 (Going-up thm for integral map)</u>

$A\subseteq B$ rings, B integral over A.

$$\begin{tikzcd}[row sep=large] B & \mathfrak{q}_1\subsetneq\cdots\subsetneq\mathfrak{q}_m \text{ chain of prime ideals of B} \\ A \arrow[u, dash] & \mathfrak{p}_1\subsetneq\cdots\subsetneq\mathfrak{p}_m\subsetneq\cdots\subsetneq\mathfrak{p}_n\ (n>m) \end{tikzcd}$$

Then the chain $\mathfrak{q}_1\subsetneq\cdots\subsetneq\mathfrak{q}_m$ can be extended to a chain

$\mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_n$ such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for $1 \le i \le n$.

<u>proof</u> By induction, we may assume $m=1, n=2$.

$$\begin{tikzcd}
B \arrow[d, dash] & \mathfrak{q}_1 & \rightsquigarrow & B/\mathfrak{q}_1 \arrow[d, dash, "\text{integral}"] & \\
A & \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 & & A/\mathfrak{p}_1 & \overline{\mathfrak{p}_2}
\end{tikzcd}$$

By 5.11 $\Rightarrow \exists$ prime ideal $\overline{\mathfrak{q}_2} \subseteq \overline{B} = B/\mathfrak{q}_1$ such that $\overline{\mathfrak{q}_2} \cap (A/\mathfrak{p}_1) = \overline{\mathfrak{p}_2}$.

Lift $\overline{\mathfrak{q}_2}$ to B, we get a prime ideal $\mathfrak{q}_2$ with the required property. □

Going-down flat or 整闭条件. 首先证 整闭是一个 local property.

<u>Prop 5.13</u> $A \subseteq B$ ring. $C =$ integral closure of A in B.
$S \subseteq A$ multi-closed.
Then $S^{-1}C$ is the integral closure of $S^{-1}A$ in $S^{-1}B$.

<u>proof</u> "S^{-1}保整性" $\Rightarrow S^{-1}C$ is integral over $S^{-1}A$.

If $\frac{b}{s} \in S^{-1}B$ integral over $S^{-1}A$ (下证 $b/s \in S^{-1}C$, 即 $\exists t$ s.t. $bt \in C$)

then $\exists$ equation $(\frac{b}{s})^n + \frac{a_1}{s_1} \cdot (\frac{b}{s})^{n-1} + \cdots + \frac{a_n}{s_n} = 0$, $a_i \in A$, $s_i \in S$.

Let $t = s_1 \cdots s_n$ and multi the equation by $(st)^n$

$\Rightarrow$ get an monic equation for bt over $A \Rightarrow bt \in C \Rightarrow \frac{b}{s} = \frac{bt}{st} \in S^{-1}C$

<u>Definition 5.14</u> A: integral domain. A is integrally closed iff A is integrally closed in Frac$A$.

(e.g. $\mathbb{Z}$ is integrally closed).

Unique factorization domain is integrally closed (类似于 $\begin{array}{c}\mathbb{Q}\\ | \\ \mathbb{Z}\end{array}$ 证明) Example 5.2

$k[x_1, \ldots, x_n]$ is integrally closed.
k field

Prop 5.15 (integral closure is a local property)

Let A be an integral domain. 以下条件等价：

(1) A is integrally closed.

(2) $A_\mathfrak{p}$ is integrally closed for each prime $\mathfrak{p}$.

(3) $A_\mathfrak{m}$ is integrally closed for each maximal ideal $\mathfrak{m}$.

proof $K = \operatorname{Frac} A$, $C :=$ integral closure of A in K.

$C_\mathfrak{m}$, $C_\mathfrak{p}$ is ~~the~~ integral closed by 5.13.

$$A \xhookrightarrow{f} C \hookrightarrow K.$$

$\forall \mathfrak{m}$

A is integrally closed $\Leftrightarrow$ f is surjective $\Leftrightarrow \forall \mathfrak{p}$, $f_\mathfrak{p}$ surjective $\Leftrightarrow f_\mathfrak{m}$ sur

$\Leftrightarrow A_\mathfrak{p}$ is int. closed $\Leftrightarrow A_\mathfrak{m}$ is

clos

Def 5.16 $A \subseteq B$ rings, $I \subseteq A$ ideal.

$x \in B$ is integral over I iff it satisfies an equation $\exists$ $x^n + a_1 x^{n-1} + \dots + a_n = 0$, $a_i \in I$.

Integral closure of I in B $= \{a \in B \mid a \text{ is integral over } I\}$

Lemma 5.17
$$\begin{tikzcd}
B \arrow[d, dash] \arrow[dr, dash] & \\
A \arrow[r, dash] & C
\end{tikzcd}$$
I, A; $C =$ integral closure of A in B. $I^e = IC =$ ext. of I in C.

Then $\boxed{\text{the integral closure of } I \text{ in } B} = \sqrt{I^e}$ (the radical of I^e)

↑ 从而 closed under addition and multiplication.

proof If $x \in B$ int. over $I \Rightarrow \exists\, x^n + a_1 x^{n-1} + \dots + a_n = 0$ $(a_i \in I)$

$\Rightarrow x \in C$ and $x^n \in I^e \Rightarrow x \in \sqrt{I^e}$.

Conversely, if $x \in \sqrt{I^e}$, then $x^n = \sum_{i=1}^{n} a_i x_i$ for some n, $a_i \in I$, $x_i \in C$.

each x_i is integral over $A \Rightarrow M = A[x_1, \dots, x_n]$ is a f.g. A-module, and

we have $x^n M \subseteq IM$.

$\Rightarrow x^n$ 满足一个系数在 I 中的方程 $\Rightarrow x^n$ is integral over I

$\Rightarrow x$ is integral over I. ▦

Prop 5.18 $A \subseteq B$ integral domain. A: integrally closed.

$x \in B$ integral over an ideal $I \subseteq A$.

Then x is alg. over $K = \mathrm{Frac}A$, and if its minimal poly over K is

$t^n + a_1 t^{n-1} + \cdots + a_n$, then $a_1, \ldots, a_n \in \sqrt{I}$ ↩ 极小多项式系数在 $\sqrt{I}$ 中.

(特别：$I = (1) = A$ 时, $a_i \in A$. 整元的极小多项式是整系数的).

proof clearly x is alg. over ~~A~~ K.

Let L/K be an ext which contains all the conjugates $x_1, \ldots, x_n$ of x.

Each x_i is integral over I. The coeff of the minimal poly of x over K are poly in the x_i.

By 5.17 $\Rightarrow a_1, \ldots, a_n$ are integral over I.

Since A is integrally closed, By 5.17 $\Rightarrow a_1, \ldots, a_n \in \sqrt{I}$. ▦

Prop 5.19 (Going-down theorem) 2024. 4月13日.

$A \subseteq B$ integral domain. A: integrally closed, B: integral over A.

$$\begin{tikzcd}[column sep=small]
B \arrow[d, dash, "\text{integral}"'] & \mathfrak{q}_1 \supsetneq \arrow[d, dash] & \mathfrak{q}_2 \supsetneq \arrow[d, dash] & \cdots \supsetneq & \mathfrak{q}_m \arrow[d, dash] & & & \\
\text{整闭 } A & \mathfrak{p}_1 \supsetneq & \mathfrak{p}_2 \supsetneq & \cdots \supsetneq & \mathfrak{p}_m \supsetneq & \cdots & \supsetneq \mathfrak{p}_n \ (n > m) &
\end{tikzcd}$$

chain of prime ideals such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i \ (1 \leq i \leq m)$.

Then the chain $\mathfrak{q}_1 \supsetneq \mathfrak{q}_2 \supsetneq \cdots \supsetneq \mathfrak{q}_m$ can be extended to a chain

$\mathfrak{q}_1 \supsetneq \mathfrak{q}_2 \supsetneq \cdots \supsetneq \mathfrak{q}_m \supsetneq \cdots \supsetneq \mathfrak{q}_n$ such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i \ (1 \leq i \leq n)$.

(注：提供另一证明)

proof may assume $m=1$ and $n=2$.

$$\begin{tikzcd}
B \arrow[rr, dash] \arrow[d, dash] & \mathfrak{q}_1 \arrow[d, dash] & B_{\mathfrak{q}_1} \\
A & \mathfrak{p}_1 \supsetneqq \mathfrak{p}_2 &
\end{tikzcd}$$

We need to show: $\mathfrak{p}_2$ is the contraction of a prime ideal in $B_{\mathfrak{q}_1}$, or equivalently,

$$\mathfrak{p}_2 B_{\mathfrak{q}_1} \cap A = \mathfrak{p}_2$$

(只要证: $\mathfrak{p}_2 B_{\mathfrak{q}_1} \cap A \subseteq \mathfrak{p}_2$)
by 5.10

(结论: $A \longrightarrow B$
$\mathfrak{p} \in \operatorname{Spec} A$ is the contraction of a prime ideal of $B \Longleftrightarrow \mathfrak{p}^{ec} = \mathfrak{p}$, i.e., $\mathfrak{p}B \cap A$ $\overset{\mathfrak{p}}{\underset{||}{}}$)

Every $x \in \mathfrak{p}_2 B_{\mathfrak{q}_1} \cap A$ is of the form $x = \frac{y}{s}$, $y \in \mathfrak{p}_2 B$, $s \in B - \mathfrak{q}_1$.

y的极小多项式系数在 $\mathfrak{p}_2$ 中
s的极小多项式系数不能在 $\mathfrak{p}_1$ 中
但 y与s的极小多项式 由 "x^{-1}" 联系, 必有 $x \in \mathfrak{p}_2$ (否则矛盾)
↑
y的极小多项
系数不能

By 5.17, y is integral over $\mathfrak{p}_2$, $\underset{\underset{yx^{-1}}{||}}{s}$ integral over A
yx^{-1} integral over A

By 5.18, the minimal equation of y over $K = \operatorname{Frac} A$ is of the form

$$y^r + u_1 y^{r-1} + \cdots + u_r = 0 \quad (u_1, \ldots, u_r \in \mathfrak{p}_2 = \sqrt{\mathfrak{p}_2}).$$

minimal equation for s over K is (因 $x^{-1} \in K$)

$$s^r + v_1 s^{r-1} + \cdots + v_r = 0 \text{ with } v_i = u_i / x^i$$

$\Rightarrow x^i v_i = u_i \in \mathfrak{p}_2 \ (1 \le i \le r)$.

But s is integral over $A \Rightarrow$ apply $I=(1)$ to 5.18, get $v_i \in A$.

Thus $x^i v_i \in \mathfrak{p}_2$ 且 $v_i \in A$.

Suppose $x \notin \mathfrak{p}_2$, then $v_i \in \mathfrak{p}_2$, $s^r \in \mathfrak{p}_2 B \subseteq \mathfrak{p}_1 B \subseteq \mathfrak{q}_1 \Rightarrow s \in \mathfrak{q}_1$ which is a contra
(与 $s \in B - \mathfrak{q}_1$ 矛盾)

Hence $x \in \mathfrak{p}_2$ and $\mathfrak{p}_2 B_{\mathfrak{q}_1} \cap A = \mathfrak{p}_2$.

補上: integral closed 的 going down 來自 Going-up + Galois property.

Thm 5.20 $A \subseteq B$ integral domains. B integral over A. A integrally closed.

(1) If B is the integral closure of A in a normal extension field L of $k = \operatorname{Frac} A$. Then any two prime ideals of B lying over the same prime $\mathfrak{p} \in \operatorname{Spec} A$ are conjugate to each other by some $\operatorname{Aut}(L/k)$. [习題课]

(2) The going down thm holds for $A \subseteq B$ in general.

proof of (2)

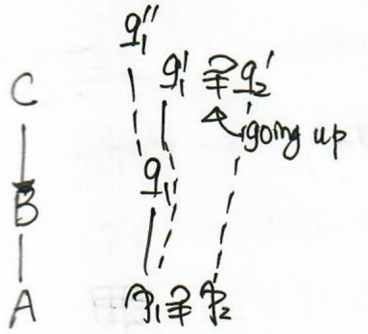
$$\begin{tikzcd}
L \arrow[d, dash] & C \arrow[d, dash] \\
L_1 = \operatorname{Frac} B \arrow[d, dash] & B \arrow[d, dash] \\
k = \operatorname{Frac} A \arrow[uu, bend left, dash, "\text{normal}"] & A
\end{tikzcd}$$

$L_1 \subseteq L/k$ normal extension containing L_1

$C =$ integral closure of A in L

$\quad =$ integral closure of B in L.

Let $\mathfrak{q}_1 \in \operatorname{Spec} B$, $\mathfrak{p}_1 = \mathfrak{q}_1 \cap A$, $\mathfrak{p}_2 \subsetneq \mathfrak{p}_1$ in $\operatorname{Spec} A$.



Take a prime ideal $\mathfrak{q}_2' \in \operatorname{Spec} C$ lying over $\mathfrak{p}_2$, and using the going-up thm for $A \subseteq C$, take $\mathfrak{q}_1' \in \operatorname{Spec} C$ lying over $\mathfrak{p}_1$ such that $\mathfrak{q}_1' \supseteq \mathfrak{q}_2'$. (但 $\mathfrak{q}_1'$ image 不一定为 $\mathfrak{q}_1$).

Let $\mathfrak{q}_1''$ be a prime ideal of C lying over $\mathfrak{q}_1$.

By (1), $\exists\, \sigma \in \operatorname{Aut}(L/k)$ such that $\sigma(\mathfrak{q}_1') = \mathfrak{q}_1''$. Put $\mathfrak{q}_2 = \sigma(\mathfrak{q}_2') \cap B$.

Then $\mathfrak{q}_1 \supseteq \mathfrak{q}_2$ and $\mathfrak{q}_2 \cap A = \sigma(\mathfrak{q}_2') \cap A \underset{(1)}{=\!=} \mathfrak{q}_2' \cap A = \mathfrak{p}_2$.

(參考 Matsumura: Commutative algebra, (5.E), Page 33) ▦

proof of (1) Let $G = \mathrm{Aut}(L/K)$.

First Assume that L/K is finite, i.e., $|G| < \infty$, $G = \{\sigma_1, \dots, \sigma_n\}$.

Let $\mathfrak{P}$ and $\mathfrak{P}'$ be prime ideals of B such that $\mathfrak{P} \cap A = \mathfrak{P}' \cap A$.

$$\begin{tikzcd} L \arrow[d, dash] & B \arrow[d, dash] & \mathfrak{P}, \mathfrak{P}' \arrow[d] \\ K & A & \mathfrak{P} \cap A \end{tikzcd}$$

Put $\sigma_i(\mathfrak{P}) = \mathfrak{P}_i$ ($\sigma_i B = B \Rightarrow \mathfrak{P}_i \in \mathrm{Spec}\, B$)

对 $\mathfrak{P}' \neq \mathfrak{P}_i$ for $i = 1, \dots, n$, then $\mathfrak{P}' \not\subseteq \mathfrak{P}_i$ (但同一个 prime ideal of A 之上的 prime ideal 无包含关系)

$\Rightarrow \exists x \in \mathfrak{P}'$ such that $x \notin \bigcup \mathfrak{P}_i = \bigcup \sigma(\mathfrak{P}_i)$ (否则 $\bigcap \mathfrak{P}_i \subseteq \mathfrak{P}'$
$\sigma_i(x) \notin \mathfrak{P}$
$\Rightarrow \mathfrak{P}_i \subseteq \mathfrak{P}'$ for some $i \Leftarrow$ prime avoidance)

Put $y = \left(\prod_i \sigma_i(x)\right)^q$ where $q = 1$ if $\mathrm{char}\, K = 0$
$q = p^\nu$ with sufficiently large ν if $\mathrm{char}\, K = p$.

Then $y \in K$.

Since A is int. closed and $y \in B \Rightarrow y \in A$.

But $y \notin \mathfrak{P}$ (for, we have $x \notin \sigma_i^{-1}(\mathfrak{P}) \Rightarrow \sigma_i(x) \notin \mathfrak{P}$)

while $y \in \mathfrak{P}' \cap A = \mathfrak{P} \cap A$. contradition.

田

对 L/K infinite, 习题.

§6 Valuation Ring (参考 T. Wedhorn: Adic Spaces and Spectral Spaces)

Definition 6.1 a totally ordered (abelian) group is a pair $(\Gamma, \le)$:

— $\Gamma \in Ab$ (whose composition law is written multiplicatively) ← Atiyah 书中写成 additive.

— "$\le$" is a total order on Γ such that: $\gamma \le \gamma' \Rightarrow \gamma\delta \le \gamma'\delta$ for all $\gamma, \gamma', \delta \in \Gamma$.

A homomorphism of totally ordered group is a homomorphism $f: \Gamma \to \Gamma'$ of groups such that for all $\gamma_1, \gamma_2 \in \Gamma$, we have "$\gamma_1 \le \gamma_2 \Rightarrow f(\gamma_1) \le f(\gamma_2)$".

Example 6.2 $\mathbb{R}^{>0}$ with (multiplication, standard order) is a totally ordered group.

$(\mathbb{R}, +)$ is a totally ordered group w.r.t the standard order.

The logarithm $\mathbb{R}^{>0} \longrightarrow \mathbb{R}$ is an isom of totally ordered groups.

Definition 6.3 A subgroup Δ of a totally ordered group Γ is called <u>convex</u> if the following equivalent conditions are satisfied for all $\delta, \delta', \gamma \in \Gamma$:

(1) $\delta \le \gamma \le 1$ and $\delta \in \Delta$ imply $\gamma \in \Delta$.

(2) $\delta, \gamma \le 1$ and $\delta\gamma \in \Delta$ imply $\delta, \gamma \in \Delta$.

(3) $\delta \le \gamma \le \delta'$ and $\delta, \delta' \in \Delta$ imply $\gamma \in \Delta$.

proof (3)⇒(1) clear.

(1)⇒(3) If $\delta(\delta')^{-1} \le \gamma(\delta')^{-1} \le 1$ and $\delta(\delta')^{-1} \in \Delta \Rightarrow \gamma(\delta')^{-1}\Delta$
but $(\delta')^{-1} \in \Delta \Rightarrow \gamma \in \Delta$.

(2)⇒(1) Let $\delta \le \gamma \le 1$ with $\delta \in \Delta$.
Then $\delta\gamma^{-1} \le 1$ and $\delta\gamma^{-1} \cdot \gamma = \delta \in \Delta \Rightarrow \gamma \in \Delta$ by assumption (2).

(1)⇒(2) If $\delta, \gamma \le 1$ and $\delta\gamma \in \Delta$, then $\delta\gamma \le \substack{\gamma \\ \delta} \le 1 \Rightarrow \gamma, \delta \in \Delta$ by assumption (1).

Example 6.4 (1) $\mathbb{R}^{>0}$ has only two convex subgroups ($\{1\}$ and $\mathbb{R}^{>0}$).

(2) Γ: totally ordered, $H<\Gamma$ subgroup. Then the convex subgroup of Γ generated by H is $\left\{ \gamma\in\Gamma \;\middle|\; \begin{array}{l} \exists h, h'\in H \\ \text{s.t } h\le\gamma\le h' \end{array} \right\}$

(3) If Δ and Δ' are two convex subgroups of Γ, then $\Delta\subseteq\Delta'$ or $\Delta'\subseteq\Delta$.

proof If $\exists\, \delta\in\Delta\setminus\Delta'$ and if $\exists\, \delta'\in\Delta'\setminus\Delta$.

After possibly replacing these element by their inverse, we may assume that $\delta, \delta'\le 1$. We may assume $\delta<\delta'<1$.

But Δ is convex and $\delta\in\Delta \Rightarrow \delta'\in\Delta$. 矛盾!

(4) If $\Gamma\xrightarrow{f}\Gamma'$ is a homomorphism of totally ordered groups, then $\ker(f)$ is a convex subgroup of Γ.

(5) Let $\Delta\subseteq\Gamma$ be a convex subgroup, and let $f:\Gamma\to\Gamma/\Delta$ be the canonical homomorphism. Then there exits a unique total order on Γ/Δ such that $f(\Gamma_{\le1}) =: (\Gamma/\Delta)_{\le1}$. Then f is a homeomorphis of totally ordered groups.

$$\gamma\in\Gamma,\quad (\Gamma/\Delta)_{\ge\gamma\Delta} = f(\Gamma_{\ge\gamma})$$

Definition 6.5 Γ: totally ordered group.

height of Γ $= ht(\Gamma) := \#\left\{ \Delta\subseteq\Gamma \;\middle|\; \begin{array}{l} \Delta\ne 1 \\ \Delta \text{ is a } \end{array} \right.$ ~~covex~~ convex subgroup$\}$

$ht(\Gamma)\in\mathbb{N}_0\cup\{\infty\}$

$ht(\mathbb{R}) = ht(\mathbb{R}^{>0}) = 1$

$ht(\Gamma)=0 \Leftrightarrow \Gamma=\{1\}$.

For Δ: convex, $ht(\Gamma)=ht(\Delta)+ht(\Gamma/\Delta)$.

Prop 6.6 Let $\Gamma \neq 1$ be a totally ordered group.

then $ht(\Gamma) = 1 \Leftrightarrow \exists$ injective homo $\Gamma \hookrightarrow \mathbb{R}_{>0}$

$\Leftrightarrow \Gamma$ is archimedean, i.e., $\forall\ \gamma, \delta \in \Gamma_{<1}$, $\exists\ m>0$ s.t. $\delta^m < \gamma$.

Definition 6.7 A: ring. A valuation of A is a map $|\cdot| : A \longrightarrow \Gamma \cup \{0\}$, where Γ is a totally ordered group, such that

(a) $|a+b| \leq \max(|a|, |b|)$ for all $a, b \in A$.

(b) $|ab| = |a| \cdot |b|$ for all $a, b \in A$

(c) $|0| = 0$ and $|1| = 1$.

If A is a topological ring, then we say $|\cdot| : A \to \Gamma \cup \{0\}$ is a continuous valuation if moreover

(d) (continuity) for all $\gamma \in \Gamma$ lying in the image of $|\cdot|$, the set $\{a \in A \mid |a| < \gamma\}$ is open in A.

Two (continuous) valuations $|\cdot|$ ($|\cdot|'$) valued in Γ (resp. Γ') are equivalent if $|a| \geq |b|$ iff $|a|' \geq |b|'$. In this case, after replacing Γ by the subgroup generated by $\mathrm{Im}(|\cdot|)$ (resp. $\mathrm{Im}(|\cdot|')$), there exists an isomorphism $\Gamma \cong \Gamma'$ such that

$$\begin{tikzcd}
 & \Gamma \cup \{0\} \arrow[d, "\cong"] \\
A \arrow[ru, "|\cdot|"] \arrow[r, "|\cdot|'"'] & \Gamma' \cup \{0\}
\end{tikzcd}$$

If $A = K$ is a field, a valuation on K is equivalent to a (surjective) homomorphism $v : K^\times \twoheadrightarrow \underset{\uparrow \text{additive}}{\Gamma}$ (put $v(0) = \infty$) such that

$$v(x+y) \geq \min\{v(x), v(y)\}$$

$$v(xy) = v(x) + v(y) \ (\Rightarrow v(1) = 0)$$

example

$|\cdot| = \rho^{v(\cdot)}$, $0<\rho<1$ fixed.

Example 6.8 (trivial valuation) Let A be a ring, $\mathfrak{p} \in \operatorname{Spec} A$. Then

$$a \in A \longmapsto \begin{cases} 1 & \text{if } a \notin \mathfrak{p} \\ 0 & \text{if } a \in \mathfrak{p} \end{cases}$$ is a valuation with value group 1.

Every valuation on A of this form is called a trivial valuation.

Definition 6.9 (Valuation Spectra) A: ring. The valuation spectrum $\operatorname{Spv}(A) = \left\{ \begin{matrix} \text{equivalence class} \\ \text{of valuation on } A \end{matrix} \right\}$ with topology generated by the subsets

$$\operatorname{Spv}(A)\left(\frac{f}{s}\right) = \{ |\cdot| \in \operatorname{Spv}(A) \mid |f| \leq |s| \neq 0 \} \quad (f, s \in A).$$

If $v \in \operatorname{Spv}(A)$, we will write $|\cdot|_v$ instead of v if we think of v as an (equivalence class of an) absolute value on A.

Can show $\operatorname{Spv}(A)$ is a spectral space.

For ring homo $A \xrightarrow{\varphi} B$, we have a continuous map ~~Spv(φ)~~ $\operatorname{Spv}(\varphi) : \operatorname{Spv} B \to \operatorname{Spv} A$, $|\cdot| \mapsto |\cdot| \circ \varphi$

Remark there is a continuous map

$$\begin{aligned} \operatorname{Supp} : \operatorname{Spv}(A) &\longrightarrow \operatorname{Spec} A \\ \underset{|\cdot|_x}{\overset{\shortparallel}{x}} &\longmapsto \operatorname{supp}(x) = \operatorname{Ker}(|\cdot|_x : A \to \Gamma_x \cup \{0\}) \end{aligned}$$

↑ this is a prime idea.

we also have a map $\operatorname{Spec} A \underset{\text{trivial valuation}}{\hookrightarrow} \operatorname{Spv}(A)$.

Example 6.10 (1) If $A = \mathbb{Q}$. Then the only valuations on $\mathbb{Q}$ are the p-adic valuations $|\cdot|_p$ for prime numbers p and the trivial valuation $|\cdot|_0$. We have $\operatorname{Spv} \mathbb{Q} = \operatorname{Spec} \mathbb{Z}$.

(2) $A=\mathbb{Z}$. Then $\mathrm{Spv}\,\mathbb{Z} = \mathrm{Spv}\,\mathbb{Q} \cup \{|\cdot|_{0,p} ;\ p \text{ prime number}\}$,

$|\cdot|_{0,p}$ is induced by the trivial valuation on $\mathbb{F}_p$ (which is a closed point)

↑ complement 是 $\mathrm{Spv}(\mathbb{Z})(\frac{p}{p})$.

2024.4.17

Recall that Let A and B be local rings with $A \subseteq B$. Then we say that B dominates A if $\mathfrak{m}_B \cap A = \mathfrak{m}_A$.

Definition 6.11 (Valuation ring) B: integral domain and $K = \mathrm{Frac}\,B$.

We say B is a valuation ring of K if the following equivalent conditions hold:

(1) For $x \neq 0$ in K, either $x \in B$ or $x^{-1} \in B$ (or both).

在6.13中我们会证 B is a local ring.

(2) ∃ totally ordered abelian group Γ, ∃ surjective homomorphism $v: K^\times \twoheadrightarrow \Gamma$

($v(0)=\infty$ 大于Γ中任一元) such that
$$v(x+y) \geq \min\{v(x), v(y)\}$$
$$v(x \cdot y) = v(x) + v(y)$$

and that $B = \{x \in K^\times \mid v(x) \geq 0\} \cup \{0\}$.

之后会证明B的maximal ideal等于$\{x \in B \mid v(x) > 0\}$.

v is called the valuation of B (and K), and $(\Gamma, \leq)$ is called the value group.

(当$\Gamma=\mathbb{Z}$, 称为离散赋值环)

rank $\Gamma = \dim B$ (此结论不证).

(3) The set of principal ideals of B is totally ordered by inclusion.

(4) The set of ideals of B is totally ordered by inclusion.

(特别: prime ideals 有全序关系: $\bullet \rightsquigarrow \bullet \rightsquigarrow \bullet \cdots \rightsquigarrow \bullet \leftarrow$ maximal/closed pt)

(5) B is local and every f.g ideal of B is principal.

(6) B is local and B is maximal for the relation of domination among local rings contained in K.

(7) $\exists$ alg. closed field L and a homomorphism $\theta : B \to L$ (not necessarily injective) w.r.t which B is maximal: if $B \subseteq B' \subseteq K$ and $\theta' : B' \to L$ extending θ, then $B = B'$.

proof (1) $\Rightarrow$ (2) Let $\Gamma = K^\times / B^\times$ (with group law written additively)

$$K^\times \xrightarrow{v} K^\times / B^\times \text{ canonical.}$$

For $\gamma, \gamma' \in \Gamma$, define $\gamma \leq \gamma' \Leftrightarrow \gamma' - \gamma \in \mathrm{Im}(B \setminus \{0\} \xrightarrow{v} \Gamma)$

(2) $\Rightarrow$ (1) clear. For $0 \neq x$, we have either $v(x) \geq 0$ or $\overset{-v(x)}{\overset{\parallel}{v(x^{-1})}} \geq 0$

(1) $\Rightarrow$ (3) Since the ordered set of principal ideals can be identified with the ordered monoid set $B - \{0\} / B^* \Rightarrow$ (3).

(3) $\Rightarrow$ (4) Suppose $I, J \subseteq B$ ideals, $I \nsubseteq J$.

We show $J \subseteq I$. Choose $a \in I \setminus J$, $b \in J$.

Since $a \notin J \Rightarrow a \notin (b)$, and hence $(b) \subseteq (a) \subseteq I$

$\Rightarrow J \subseteq I$.

(4) $\Rightarrow$ (5) Since the set of ideals of ~~R~~ B is totally ordered, ~~R~~ B has a unique maximal ideal $\Rightarrow$ ~~R~~ B is local.

To prove that every f.g ideal is principal, it is suffices to show that any ideal which is generated by two elements is in fact principal. But if $I = (f, g)$, either $(f) \subseteq (g)$ or $(g) \subseteq (f)$, hence $I = (f)$ or (g).

(5)⇒(1) Suppose $a, b \in B\setminus\{0\}$ (下证 $\frac{a}{b}$ or $\frac{b}{a} \in B$).

$I=(a,b)$. Then I is principal, I/mI is a one-dimensional Vector space over $k=R/m$ ⇒ images of a and b are linearly dependent over k.

⇒ $\exists u, v \in B$ s.t $ua+vb = mI$ with u, v 不全在 m 中

⇒ $ua+vb = xa+yb$ with $x, y \in m$

⇒ $a(u-x) = b(y-v)$

Now if (不妨设 $u \notin m$) u is a unit, so is $u-x$ ⇒ $\frac{a}{b} = \frac{y-v}{u-x} \in B$.

(1)⇒(6) Suppose $B \subseteq B' \subseteq K$ with B' local (若 $B \neq B'$, 下证 B' does not dominate B)

If $x \in B'$ and $x \notin B$, then $x^{-1} \in B \subseteq B'$

⇒ x is a unit in B' (but x is not a unit in B)

⇒ B' does not dominate B.

(6)⇒(7) $k = B/m$ residue field. $k \to \bar{k}$ alg. closure.

$\theta : B \to k \to \bar{k}$.

Suppose
$$\begin{tikzcd} B \arrow[r, phantom, "\subseteq"] \arrow[d, "\theta"'] & B' \arrow[r, phantom, "\subseteq"] \arrow[dl, dashed, "\theta'"] & K \\ \bar{k} & & \end{tikzcd}$$

Let $m' = \mathrm{Ker}(\theta')$ (m' is a prime)

⇒ θ' factor through the localization $B'' = B'_{m'}$.

So by replacing B' by B'', and we may assume that B' is local with maximal ideal m'. Since θ' extending θ, m maps to m'

⇒ B' dominate B ⇒ $B' = B$ by assumption on (6).

(7)⇒(1) 采用书中证明(待会记). □

实际给出了一种构造 valuation ring 的方法.

先讨论几个例子

6.12 Examples of valuation rings

(1) dim = 1 的 valuation ring.

$\mathbb{Z}_p \subseteq \mathbb{Q}_p$ p-adic valuation ring/field.

$$\widehat{\bigcup_{n\to\infty}\mathbb{Q}_p(p^{1/p^n})}^{\,p\text{-adic}}$$

(2) dim ≥ 2

$R = k[x,y]$, k field.

$v : k(x,y) \twoheadrightarrow \mathbb{Z}^2$

$v(x) = (1,0) \leq v(y) = (0,1)$

v(polynomial) = minimal values among those of its monomials

(3) $k[x] \subseteq k[x^{1/2}] \subseteq \cdots \subseteq k[x^{1/2^n}] \subseteq \cdots$

x : transcendental over the field k.

$\mathcal{O}_n = k[x^{1/2^n}]_{P_n}$ d.v.r, $P_n = (x^{1/2^n})$ prime ideal.

But $P_{n+1} \cap k[x^{1/2^n}] = P_n$

$\Rightarrow \mathcal{O}_n \subseteq \mathcal{O}_{n+1}$

$\mathfrak{m}_{n+1} \cap \mathcal{O}_n = \mathfrak{m}_n$

$\Rightarrow \mathcal{O} = \bigcup_n \mathcal{O}_n$ is a non-noether valuation ring of the field $k(x^{1/2^n}, n \in \mathbb{N})$. The value group is order isomorphic to the subgroup $\{\frac{z}{2^n} \mid z \in \mathbb{N}, n \in \mathbb{N}\} \subseteq \mathbb{Q}$.

Hence this example yields a non-noetherian valuation ring of Krull dimension 1.

Proposition 6.13 Let B be a valuation ring. Then

(一) B is a local ring.

proof $\mathfrak{m} \overset{\text{def}}{=} \left\{ \begin{matrix} \text{non-units} \\ \text{in } B \end{matrix} \right\}$. We show B is an ideal (hence must be maximal ideal)

First, $x \in \mathfrak{m} \Leftrightarrow x = 0$ or $x^{-1} \notin B$. in Frac B

If $a \in B$ and $x \in \mathfrak{m}$, then $ax \in \mathfrak{m}$ (否则 $(ax)^{-1} \in B \Rightarrow \overset{a\cdot(ax)^{-1}}{\overset{\shortparallel}{x^{-1}}} \in B$ 矛盾)

Let $x, y \in \mathfrak{m} \setminus \{0\}$, then either $xy^{-1} \in B$ or $x^{-1}y \in B$.

If $xy^{-1} \in B$, then $x + y = (1 + xy^{-1})y \in \mathfrak{m}$

If $x^{-1}y \in B$, then $x + y \in \mathfrak{m}$.

$\Rightarrow \mathfrak{m}$ is an ideal, which is maximal $\Rightarrow$ B is a local ring.

(二) 若 B' is a ring such that $B \subseteq B' \subseteq K$, then B' is a valuation ring of K.

(利用 B' is a valuation ring $\Leftrightarrow \forall 0 \neq x \in K$, $x \in B'$ or $x^{-1} \in B'$).

(三) B is integrally closed in K.

proof Let $x \in K$ be integral over B such that $x^n + b_1 x^{n-1} + \cdots + b_n = 0$ ($b_i \in B$).

If $x \in B$, OK.

If not, then $x^{-1} \in B \Rightarrow x = -(b_1 + b_2 x^{-1} + \cdots + b_n x^{(1-n)}) \in B$.

$\square$

(四) 习题 B valuation ring. Then for all $\mathfrak{p} \in \operatorname{Spec} B$, $B/\mathfrak{p}$ and $B_{\mathfrak{p}}$ are valuation rings.

Construction 6.14 (构造 valuation ring)

K : any field, Ω : algebraically closed field.

$$\Sigma = \left\{ (A, f) \;\middle|\; \begin{matrix} A \subseteq K \text{ subring.} \\ f : A \longrightarrow \Omega \text{ homomorphism} \end{matrix} \right\}.$$

Σ has a partially ordered structure:

$$(A,f) \leq (A',f') \iff A \subseteq A' \text{ and } f'|_A = f.$$

Zorn's lemma: Σ has at least one maximal element, Say $(B,g) \in \Sigma$

We will show: B is a ~~valuation~~ valuation ring.

<u>Lemma 6.15</u> B is a local ring with maximal ideal $\mathfrak{m} = \ker(B \xrightarrow{g} \Omega)$

<u>proof</u> $g: B \to \Omega$, $g(B)$ integral domain $\Rightarrow \ker g = \mathfrak{m}$ is a prime ideal

We can extend g to a homomorphism $\bar{g}: B_{\mathfrak{m}} \to \Omega$ by

$\bar{g}\left(\frac{b}{s}\right) = g(b)\,g(s)^{-1}$

$(\underset{\text{unit}}{\underset{\Uparrow}{s}} \in B \setminus \mathfrak{m} \Rightarrow \underset{\text{unit}}{\underset{\Uparrow}{g(s)^{-1}}}$ make in B

Since (B,g) maximal $\Rightarrow B = B_{\mathfrak{m}}$

$\Rightarrow$ B is a local ring with maximal ideal $\mathfrak{m}$.

后面为证 B 是 valuation ring, 要证 $x \in K\setminus\{0\}$, $x \in B$ or $x^{-1} \in B$.

<u>Lemma 6.16</u> Let $x \in K\setminus\{0\}$, $B[x] \subseteq K$ subring generated by x over B.

$\mathfrak{m}[x] = \mathfrak{m}(B[x]) = (\text{extension of } \mathfrak{m} \text{ in } B[x])$.

Then either $\mathfrak{m}[x] \neq B[x]$ or $\mathfrak{m}[x^{-1}] \neq B[x^{-1}]$.

<u>proof</u> Suppose that $\mathfrak{m}[x] = B[x]$ and $\mathfrak{m}[x^{-1}] = B[x^{-1}]$.

then we have equations:

$$(6.16.1) \qquad 1 = u_0 + u_1 x + \dots + u_m x^m \quad (u_i \in \mathfrak{m})$$

$$(6.16.2) \qquad 1 = v_0 + v_1 x^{-1} + \dots + v_n x^{-n} \quad (v_i \in \mathfrak{m})$$

We may suppose that the degrees m and n are as small as possible.

Suppose that $m \geq n$, and multiply (6.16.2) by x^n:

$$(6.16.3) \qquad (1-v_0)x^n = v_1 x^{n-1} + \dots + v_n$$

Since $v_0 \in \mathfrak{m} \Rightarrow 1 - v_0$ is a unit in B by lemma 6.15.

we may write (6.16.3) in the form: $x^n = w_1 x^{n-1} + \dots + w_n$ $(w_j \in \mathfrak{m})$.

Hence we may replace x^m in (6.16.1) by $w_1 x^{m-1} + \dots + w_n x^{m-n}$, and this contradicts the minimality of $\mathfrak{m}$.

Lemma 6.17 B is a valuation ring with fraction field K.

proof We have to show: if $0 \neq x \in K$, then either $x \in B$ or $x^{-1} \in B$.

By Lemma 6.16, we may assume $m[x] \neq B' := B[x]$ is not the unit ideal.

Then $m[x] \subseteq m' \subseteq B'$ for some maximal ideal $m' \subseteq B'$.

(以下要证: $x \in B$, 方式为: $g: B \to \Omega$ 可以拓到 $B[x] \to \Omega$).

We have $m' \cap B = m$ (since $m' \cap B$ is a proper ideal of B and contains the maximal ideal m).

$\Rightarrow B \hookrightarrow B'$ induces $k = B/m \hookrightarrow k' = B'/m'$,

and $k' = k[\bar{x}]$, $\bar{x}$ = image of x in k'.

$\Rightarrow \bar{x}$ is alg over k and k' is a finite alg. ext of k.

Now $g: B \to \Omega$ induces an embedding $\bar{g}: k \hookrightarrow \Omega$ (since $m = \ker g$)

Ω ↑ 代数闭

Composing $\bar{g}$ with

$$\begin{tikzcd} B' \arrow[r] \arrow[rr, bend right, "g'"'] & k' \arrow[r, hook, "\bar{g}", "代数闭"'] & \Omega \end{tikzcd}$$

we have $g': B' \to \Omega$ extending g.

Since (B, g) is maximal $\Rightarrow B = B'$ and $x \in B$. ▤

Thm 6.18 $A \subseteq K$ subring, K: field. $\bar{A}$ = integral closure of A in K. Then $\bar{A} = \bigcap_{A \subseteq B \subseteq K} B$, B: valuation ring of K

proof Let $B \subseteq K$ be a valuation ring such that $A \subseteq B$.

Since B is integrally closed $\Rightarrow \bar{A} \subseteq B$. $\left(\bar{A} \subseteq \bigcap_{A \subseteq B \subseteq K} B\right)$

Conversely, let $x \notin \bar{A}$, then $x \notin A' := A[x^{-1}]$. (we show: $\exists B$ such that $x \notin B$)

$\Rightarrow x^{-1} \in A[x^{-1}] = A'$ is a non-unit in A' $\Rightarrow x^{-1}$ is contained in a maximal ideal $m' \subseteq A'$.

$\Rightarrow x^{-1}$ is contained in a maximal ideal $m' \subseteq A'$

Let Ω be an alg. closure of the field $k' = A'/m'$

Then $A \hookrightarrow A' \twoheadrightarrow k'$ defines a homomorphism $A \to \Omega$.

By 6.17, we can extend $A \to \Omega$ to some valuation ring B:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \Omega \\ B \arrow[ur] & \end{tikzcd}$$

Since x^{-1} maps to zero in $k' = A'/m'$ $\Rightarrow x \notin B$ (否则在 B 中 $1 = x \cdot x^{-1}$, x^{-1} 不能为 0)

zero in Ω

Prop 6.19 $A \subseteq B$ integral domain, B: f.g. over A.

$v \in B \setminus \{0\}$. Then $\exists\, 0 \neq u \in A$ satisfies the following property:

(*) any homo $A \xrightarrow{f} \Omega$ into an alg. closed field Ω such that $f(u) \neq 0$ can extended to a homomorphism $g: B \to \Omega$ such that $g(v) \neq 0$ (特别可取

先看一个推论（之前在Noether那里没证）.

Corollary 6.20 (One form of Hilbert's Nullstellensatz)

k field. B: f.g. k-algebra. If B is a field, then it is a finite alg.

of k.

proof In prop 6.19, take $A = k$, $v = 1$, $\Omega =$ alg. closure of k.

所有延拓 $k \hookrightarrow \Omega$ 都是~~单射~~, $B \hookrightarrow \Omega$ 中 B/k 也是代数扩张, B

$$\begin{tikzcd} k \arrow[r, hook] \arrow[d] & \Omega \\ B \arrow[ur] & \end{tikzcd}$$

k上都是整的, 从而 B/k finite as k-module.

proof of Prop 6.19 By induction on the number of generators of B/A, we may

$B = A[x]$.

(1) If x is transcendental over A, i.e., that no-non-zero polynomial with coe in A has a root. Let $v = a_0x^n + a_1x^{n-1} + \cdots + a_n$ and take $u = a_0$.

then if $A \xrightarrow{f} \Omega$ such that $f(u) \neq 0$, then $\exists\, \xi \in \Omega$ such that $f(a_0)\xi^n + f(a_1)\xi^{n-1} + \cdots + f(a_n) \neq 0$ (since Ω is

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & \Omega \\ B = A[x] \arrow[ur, dashed, "g"] & \end{tikzcd}$$

then define $g: B \to \Omega$ by $g(x) = \xi$. $\rightsquigarrow g(v) \neq 0$.

(2) Now assume x is alg. over A (hence alg over Frac$A$).

v is a polynomial in $x \Rightarrow v^{-1}$ is alg. over Frac$A$.

$$\text{assume} \begin{cases} a_0 x^m + a_1 x^{m-1} + \cdots + a_m = 0 \ (a_i \in A) & (*1) \\ a_0' v^{-n} + a_1' v^{1-n} + \cdots + a_n' = 0 \ (a_j' \in A) & (*2). \end{cases}$$

Let $u = a_0 a_0'$ and $f: A \longrightarrow \Omega$ such that $f(u) \neq 0$.

Then f can be extended to $f_1: A[u^{-1}] \longrightarrow \Omega$ with $f_1(u^{-1}) = f(u)^{-1}$.

Apply 6.17 to get $h: C \longrightarrow \Omega$, C: valuation ring containg $A[u^{-1}]$.

By assumption (*1), x is integral over $A[u^{-1}] \Rightarrow x \in C$ (由于 C int. closed)

$\Rightarrow \overset{A[x]}{\overset{\|}{B}} \subseteq C$ and $\underset{\text{poly. in } x.}{\underset{\|}{v}} \in C$.

By (*2), v^{-1} is integral over $A[u^{-1}] \Rightarrow v^{-1} \in C \Rightarrow v$ is a unit in C.
▦

hence $h(v) \neq 0$. Now take $g = h|_B$.

Definition 6.21 (Discrete valuation ring)

K: field. a discrete valuation on K is a mapping $v: K^* \longrightarrow \mathbb{Z}$ such that

(1) v surjective

(2) $v(xy) = v(x) + v(y)$ (v is a homomorphism)

(3) $v(x+y) \geqslant \min\{v(x), v(y)\}$.

此时称K是 d.v.f.

Often extend v to $v: K \longrightarrow \mathbb{Z}$ by $v(0) = +\infty$.

Set $\mathcal{O}_v = \{x \in K \mid v(x) \geqslant 0\}$, which is called the valuation $\underline{\underline{\text{ring}}}$ of K

(of $\underset{\text{height}}{\underline{\text{rank 1}}}$)

<u>Example 6.22</u> (1) $K=\mathbb{Q}$. p: prime number. Any $0\neq x\in\mathbb{Q}$ can be written as $x=p^a y$ (a

y的分母与p互素). Define $v_p(x)=a$. Then $\mathcal{O}_v=\mathbb{Z}_{(p)}=\{\frac{a}{b} \mid p\nmid b\}$.

(2) $K=k(x)$, k field, x: indeterminate.

Take a fixed irr. polynomial $f\in k[x]$ with v_f similar to (1).

Then $\mathcal{O}_v=k[x]_{(f)}$.

<u>Definition 6.23</u> An integral domain A is a discrete valuation ring iff $\exists$ discrete valuation ~~ring~~ v on $K=\mathrm{Frac}A$ s.t. $A=\mathcal{O}_v$.

Now A is a local ring (integrally closed) with maximal ideal $\mathfrak{m}=\{x\in K \mid v(x)>$

units $=A-\mathfrak{m}=\{x\in K \mid v(x)=0\}$.

特别, 对 $x,y\in A$. 若 $v(x)=v(y)$, then $v(xy^{-1})=0$, 从而 xy^{-1} is a unit

$\Rightarrow (x)=(y)$.

<u>以下确定 d.v.r A 中的 ideals.</u>

若 $0\neq I\subseteq A$ ideal. $\Rightarrow \exists$ least integer k s.t. $v(x)=k$ for some $x\in I$

$\{k=\min\{v(x) \mid x\in I\}\}$.

$\Rightarrow I=\{y\in A \mid v(y)\geq k\}$ (对 $y\in I$, $v(y)=v(x)+v(\cdots)$

$\Rightarrow y=x\cdot(\cdots)\in$ 右边)

"$\subseteq$" clear

"$\supseteq$" $\forall v(y)\geq k \Rightarrow v(yx^{-1})\geq 0 \Rightarrow yx^{-1}\in A \Rightarrow y\in Ax\subseteq I$

The only ideals $\neq 0$ in A are the ideals $\mathfrak{m}_k=\{y\in A \mid v(y)\geq k\}$.

these form a single chain $\mathfrak{m}\supseteq\mathfrak{m}_1\supseteq\mathfrak{m}_2\supseteq\cdots$

$\Rightarrow A$ is Noetherian (往上走有限步会 stop)

Moreover, $v: K^\times \longrightarrow \mathbb{Z}$ surjective $\Rightarrow \exists\, \pi \in m$ such that $v(\pi)=1$

这种元素称为 uniformizer.

then $m=(\pi)$ and $m_k = (\pi^k) = m^k$.

Hence m is the only non-zero prime ideal of A, and A is thus a Noetherian local domain of dim 1, in which every non-zero ideal is a power of the maximal ideal.

反之有 其它1维 valuation ring 都是非 Noether.

Prop 6.24 A: Noether local domain of dim 1, $m \subseteq A$ maximal ideal

$k = A/m$.

TFAE
(1) A is a d.v.r. ⤹ 之前证过 valuation ring 都整闭. Prop 6.13 (3)
(2) A is integrally closed
(3) m is a principal ideal
(4) $\dim_k m/m^2 = 1$.
(5) Every non-zero ideal is a power of m.
(6) $\exists x \in A$ s.t every non-zero ideal is of the form (x^k), $k \geq 0$.

Two facts are known

(A) If I is an ideal $\neq (0),(1)$, then $\exists\, n$, s.t $m^n \subseteq I$.

原因: $\sqrt{I} = \bigcap_{\substack{I \subseteq p \\ \text{prime}}} p = m$ (dim $A=1$, m 是 A 中仅有的 prime ideal)

但 A Noether $\Rightarrow m^N \subseteq I$ for some N
m f.g

(B) $m^n \neq m^{n+1}$ $(\forall n \geq 0)$. [否则 $m^n = 0$, Not a domain].

Pf (1)⇒(2) 已证.

(2)⇒(3) Let $0\neq a\in \mathfrak{m}$. By (A) ⇒ $\exists N$ s.t $\mathfrak{m}^N\subseteq(a)$
$\mathfrak{m}^{N-1}\nsubseteq(a)$.

Choose $b\in \mathfrak{m}^{N-1}$ s.t $b\notin(a)$. Let $x=a/b\in K=\mathrm{Frac}A$.

Since $b\notin(a)\Rightarrow x^{-1}=\frac{b}{a}\notin A\Rightarrow x^{-1}$ is not int over A

$\Rightarrow x^{-1}\mathfrak{m}\nsubseteq\mathfrak{m}$ (如果 $x^{-1}\mathfrak{m}\subseteq\mathfrak{m}$, $\mathfrak{m}$ could be a faithful $A[x^{-1}]$ module which is f.g as an A-module ⇒ x^{-1} integral over A).

但 $x^{-1}\mathfrak{m}\subseteq A\Rightarrow \mathfrak{m}=Ax=(x)$.
$(\mathfrak{m}\subseteq Ax)$

(3)⇒(4). By Nakayama ⇒ $\dim_k \mathfrak{m}/\mathfrak{m}^2\le 1$.
By (B) ⇒ $\mathfrak{m}/\mathfrak{m}^2\neq 0\Rightarrow\dim_k\mathfrak{m}/\mathfrak{m}^2=1$.

(5)⇒(6). By (B), $\mathfrak{m}\neq\mathfrak{m}^2\Rightarrow\exists x\in\mathfrak{m}, x\notin\mathfrak{m}^2$.
But $(x)=(\mathfrak{m}^r)$ by hypothesis ⇒ $r=1$, $(x)=\mathfrak{m}$, $(x^k)=\mathfrak{m}^k$.

(4)⇒(5). By (A), for all ideal $I\neq(0),(1)$, $I\supseteq\mathfrak{m}^N$ for some N.
Now for the Artin ring $A/\mathfrak{m}^N$ ⇒ every ideal in $A/\mathfrak{m}^N$ is principal $\Leftarrow\dim\mathfrak{m}/\mathfrak{m}^2\le 1$
⇒ I is a power of $\mathfrak{m}$.

(6)⇒(1). $\mathfrak{m}=(x)$. By (B), $(x^k)\neq(x^{k+1})$.
Hence if $0\neq a\in A$, we have $(a)=(x^k)$ for exactly one value of k.
Define $v(a)=k$, and extend v to K^* by $v(ab^{-1})=v(a)-v(b)$.
Can check v is well-defined and is a d.v.

Dedekind Domain (global version of d.v.r)

Thm 6.25 A : Noether domain of dim 1. TFAE:

(1) A is integrally closed.

(2) Every local ring $A_{\mathfrak{p}}$ $(\mathfrak{p} \neq 0)$ is a d.v.r ($\Leftrightarrow$ integrally closed).

Such rings are called Dedekind domain.

In a Dedekind domain, every non-zero ideal has a unique factorization a product of prime ideals.

Example 6.26

(1) Every principal ideal domain A is a Dedekind domain

$\Rightarrow$ A is Noetherian since every ideal is f.g.

A is of dimension 1

Every local ring $A_{\mathfrak{p}}$ $(\mathfrak{p} \neq 0)$ is a principal ideal domain, hence a d.v.r.

(2) $K/\mathbb{Q}$ alg. number field ($K/\mathbb{Q}$ finite ext)

its ring of integers A = int. closure of $\mathbb{Z}$ in K.

Then A is a Dedekind domain.

首先A是 integrally closed 且 Noetherian ($\exists v_1, \dots, v_n$, s.t $A \subseteq \mathbb{Z}v_1 + \dots + \mathbb{Z}v_n$ A is f.g. $\mathbb{Z}$-module)

下证明: $\forall$ non-zero prime ideal $\mathfrak{p} \subseteq A$ is maximal (hence dim A=1)

因 $\mathfrak{p} \cap \mathbb{Z} \neq 0$, $\mathfrak{p} \cap \mathbb{Z}$ is a maximal ideal of $\mathbb{Z}$ $\Rightarrow$ $\mathfrak{p}$ is a maximal ideal of A

$$\begin{tikzcd} A \arrow[d, dash, "f.g"] & \mathfrak{p} \\ \mathbb{Z} & \mathfrak{p} \cap \mathbb{Z} \text{ maximal} \end{tikzcd}$$

(**Recall**

$$\begin{tikzcd} B \arrow[d, dash, "\text{integral}"] & \mathfrak{q} \arrow[d, dash] \\ A & \mathfrak{p} \end{tikzcd}$$

B : field $\Leftrightarrow$ A field

$\mathfrak{q}$ maximal $\Leftrightarrow$ $\mathfrak{p}$ maximal)

4月24日

Lemma 6.27 A: integrally closed domain with $K = \mathrm{Frac}(A)$

$$\begin{tikzcd} B \arrow[d, dash] & L \arrow[d, dash] \\ A & K \end{tikzcd}$$

L/K finite separable alg. ext

B: integral closure of A in L.

Then $\exists$ basis $v_1, \dots, v_n$ of L over K such that $B \subseteq \sum_{j=1}^{n} A v_j$.

proof If $v \in L$, then v is alg. over K, and therefore satisfies an equati

of the form $a_0 v^r + a_1 v^{r-1} + \dots + a_n = 0 \; (a_i \in A)$.

$\Rightarrow a_0 v =: u$ integral over A

Thus $a_0 v = u \in B$

Thus, given any basis of L/K, we may multiply the basis by suitable eleme

of A, then get a basis $u_1, \dots, u_n$ such that $u_i \in B \; (\forall i)$.

Let $\mathrm{Tr}: L \longrightarrow K$ be the trace

L/K separable $\Rightarrow L \times L \xrightarrow{\mathrm{tr}(xy)} K$ is non-degenerate.

Let $v_1, \dots, v_n$ be the dual basis such that $\mathrm{Tr}(u_i v_j) = \delta_{ij}$.

Now for $x \in B$, $x = \sum_j x_j v_j \; (x_j \in K)$. 下证 $x_i \in A$.

We have $x u_i \in B \; (u_i \in B) \Rightarrow \mathrm{Tr}(x u_i) \in A$. trace of an element is a multiple of one of the coeff in the minima polynomial.

$\Rightarrow \mathrm{Tr}(x u_i) = \sum_j \mathrm{Tr}(x_j u_i v_j) = \sum_j x_j$

$\mathrm{Tr}(u_i v_j) = \sum_j x_j \delta_{ij} = x_i$

$\Rightarrow x \in A \Rightarrow B \underset{j}{\subseteq} A v_j$. ∎

讨论 class group

<u>Definition 6.28</u> A: integral domain, $K = \mathrm{Frac} A$. $M \subseteq K$ sub-A-module.

We call M a fractional ideal of A if $xM \subseteq A$ for some $x \neq 0$ in A.

— ordinary ideals (integral ideals) are fractional ideals.

— principal fractional ideals are generated by some element $u \in K$, denoted by (u) or Au.

For a fractional ideal $M \subseteq K$, we define $(A:M) = \{x \in K \mid xM \subseteq A\}$

<u>Some easy facts</u>

① If $M \subseteq K$ is a f.g. A-module, then M is a fractional ideal.

($M = Ax_1 + \cdots + Ax_n$, with $x_i = \frac{y_i}{\cancel{z_i}}$ $(1 \le i \le n)$, $y_i \in A$, $\cancel{z_i} \in A$
令 $\prod z_i = z$ then $zM \subseteq A$)

② If A is Noetherian, then every fractional ideal is f.g.

In fact, if $M \subseteq K$ is a fractional ideal and if $zM \subseteq A$, then zM is f.g ideal and $M = z^{-1}I$.

③ $M \subseteq K$ sub-A-module. Say M is an invertible ideal if $\exists N \subseteq K$ submod such that $MN = A$.

In this case, $N = (A:M)$ and M is f.g. fractional ideal.

<u>Indeed</u>, $N \subseteq (A:M) = (A:M)MN \subseteq AN = N$

因为 $M \cdot (A:M) = A \Rightarrow \exists \sum x_i y_i = 1$, $x_i \in M$, $y_i \in (A:M)$ $(1 \le i \le n)$.

$\Rightarrow \forall x \in M$, $x = \sum (y_i x) x_i$, $y_i x \in A$

$\Rightarrow M$ is generated by $x_1, \cdots, x_n$.

(4) Every non-zero principal fractional ideal (u) is invertible, its inverse is (u^{-1}).
The invertible ideals form a group w.r.t multiplication, whose identity element in A is (1).

Invertibility is a local property.

<u>Prop 6.29</u> $M \subseteq K$ fractional ideal. 以下等价:

(1) M is invertible.

(2) M is f.g and for each prime ideal $\mathfrak{p}$, $M_{\mathfrak{p}}$ is invertible.

(3) M is f.g and $\forall$ maximal ideal $\mathfrak{m}$, $M_{\mathfrak{m}}$ is invertible.

<u>proof</u> (1)⇒(2) $A_{\mathfrak{p}} = (M\cdot(A:M))_{\mathfrak{p}} = M_{\mathfrak{p}}\cdot(A:M)_{\mathfrak{p}} \xlongequal{M:f.g} M_{\mathfrak{p}}\cdot(A_{\mathfrak{p}}:$

(2)⇒(3) Clear

(3)⇒(1). Let $I = M\cdot(A:M)$ ideal of A.

$\forall \mathfrak{m} \in Max(A)$, $I_{\mathfrak{m}} = A_{\mathfrak{m}}$. $\Rightarrow I \not\subseteq \mathfrak{m}$ $\forall \mathfrak{m} \in Max(A)$

$\Rightarrow I = A$ and M is invertible. ∎

<u>Prop 6.30</u> A: local domain. Then A is a d.v.r $\Leftrightarrow$ every non-zero frac ideal of A is invertible

<u>proof</u> $\Rightarrow$ Let $\mathfrak{m} = (x)$ be the maximal ideal of A.

Let $M \neq 0$ be a fractional ideal.

Then $\exists y \neq 0$ in A s.t $yM \subseteq A$, yM usual ideal, he of the form $yM = (x^r)$.

$\Rightarrow M = (x^{r-s})$ where $s = v(y)$.

$\Leftarrow$ Every non-zero ~~non-z~~ ideal is invertible, hence f.g $\Rightarrow A$ is Noether.

It is enough to show: every non-zero integral ideal is a power of $\mathfrak{m}$.

If not, let $\Sigma = \left\{ \begin{matrix}\text{non-zero ideals} \\ \text{which are not power of } \mathfrak{m}\end{matrix} \right\}$. $I \in \Sigma$ maximal element.

Then $I \neq \mathfrak{m}$, hence $I \subsetneq \mathfrak{m} \Rightarrow \mathfrak{m}^{-1} I \subsetneq \mathfrak{m}^{-1}\mathfrak{m} = A$ is a proper ideal and $I \subseteq \mathfrak{m}^{-1} I$.

If $\mathfrak{m}^{-1} I = I$, then $I = \mathfrak{m} I \Rightarrow I = 0$ by Nakayama.

Hence $I \subsetneq \mathfrak{m}^{-1} I$ and $\mathfrak{m}^{-1} I$ is a power of $\mathfrak{m}$ by the maximality of I

$\Rightarrow I$ is also a power of $\mathfrak{m}$. 矛盾! ∎

Global version of previous prop 6.30!

Thm 6.31 A: integral domain. Then A is a Dedekind domain iff every non-zero fractional ideal of A is invertible.

proof $\Rightarrow$ Let $M \neq 0$ be a fractional ideal. Since A is Noether $\Rightarrow M$ is f.g.

$\forall \mathfrak{p} \in \operatorname{Spec} A$, $M_\mathfrak{p}$ is a fractional ideal $\neq 0$ of the d.v.r $A_\mathfrak{p}$

$\Rightarrow M_\mathfrak{p}$ is invertible $\forall \mathfrak{p}$ $\Rightarrow M$ is invertible.

$\Leftarrow$ Every non-zero integral ideal is invertible, hence f.g

$\Rightarrow$ A is Noether.

We show: each $A_\mathfrak{p}$ $(\mathfrak{p} \neq 0)$ is a d.v.r.

by prop 6.30, enough to show: each integral ideal $\neq 0$ in $A_\mathfrak{p}$ is invertible.

Let $J \neq 0$ be an integral ideal of $A_\mathfrak{p}$. Let $I = J^c = J \cap A$.

$\Rightarrow I$ is invertible $\Rightarrow J = I_\mathfrak{p}$ is invertible. ∎

Corollary 6.32 If A is a Dedekind domain, the non-zero fractional ideals form a group w.r.t multiplication. This group is called the group of ideals denoted by $\mathcal{I}$.

(由于A中 ideal可写成 prime的 products, 因此 $\mathcal{I}$ is a free abelian group generated by the non-zero prime ideals of A).

Remark 6.33 If I is a non-zero fractional ideal of a Dede domain R, then I can be factored uniquely as $\mathfrak{P}_1^{n_1} \cdots \mathfrak{p}_{r}^{n_r}$ consequently, the non-zero fractional ideals form a group un multiplication.

Remark 6.34 K = Frac A with A Dedekind domain.

We have a group homomorphism

$$\phi : K^* \longrightarrow I$$
$$u \longmapsto (u)$$

$P = \mathrm{Im}(\phi)$ = group of principal fractional ideals.

$H = I/P$ called the ideal class group of A.

$U = \mathrm{Ker}(\phi) = \{u \in K^* \mid (u) = (1)\}$ = group of units of A.

We have an exact sequence

$$1 \longrightarrow U \longrightarrow K^* \longrightarrow I \longrightarrow H \longrightarrow 1.$$

补. Kähler differentials A

Definition A.1 A: ring. B: A-algebra. M: B-module. An A-derivation of B into M is an A-linear map $d : B \to M$ such that the Leibniz rule

$$(*) \quad d(b_1 b_2) = b_1 db_2 + b_2 db_1 \quad \forall b_1, b_2 \in B$$

is satisfied and that $da = 0 \ \forall a \in A$. ("elements of A are constant")

$$\Updownarrow$$

$$\left(\text{由 } (*),\ d(1) = 0 \text{ 且 } d(a) = d(a \cdot 1) \underset{\substack{\uparrow \\ A\text{-linear}}}{=} a \cdot d1 = 0 \right)$$

$$\mathrm{Der}_A(B, M) = \{ A\text{-derivation of } B \text{ into } M \}$$

Definition A.2 B: A-algebra. The module of relative differential forms of B over A is a B-module $\Omega^1_{B/A}$ endowed with an A-derivation $d : B \to \Omega^1_{B/A}$ satisfying the following universal property:

— For any B-module M and for any A-derivation $d' : B \to M$, there exists a unique homomorphism of B-modules $\phi : \Omega^1_{B/A} \to M$ such that $d' = \phi \circ d$

$$\begin{tikzcd}
B \arrow[r, "d'"] \arrow[d, "d"'] & M \\
\Omega^1_{B/A} \arrow[ur, dashed, "\exists! \phi"'] &
\end{tikzcd}$$

Prop A.3 The module of relative differential forms $(\Omega^1_{B/A}, d)$ exists and is unique up to unique isom.

In particular, for any B-module M, $\mathrm{Hom}_B(\Omega^1_{B/A}, M) \xrightarrow{\phi \ \mapsto \ \phi \circ d} \mathrm{Der}_A(B, M)$ is an isom.

proof The uniqueness follows from the definition.

show existence F: free B-module generated by the symbols db (b

$$E = F \Big/ \left\langle \left\{ \begin{array}{l} da \\ d(b_1+b_2)-db_1-db_2 \\ d(b_1b_2)=b_1db_2+b_2db_1 \end{array} \right| \begin{array}{l} a\in A \\ b_1,b_2\in B \end{array} \right\} \right\rangle$$

$d: B \longrightarrow \Omega^1_{B/A} = F/E$ (by construction, $\Omega^1_{B/A}$ is gen as a B-module by dB

$b \longmapsto \underset{db}{\underset{\|}{\underline{\text{image of } b \text{ in } F/E}}}.$

Can check $(\Omega^1_{B/A}, d)$ has the required properties. □

Example A.3 A: ring. $B = A[T_1, \cdots, T_n]$. Then $\Omega^1_{B/A}$ is the free B-module generated by the symbol dT_i.

— Let $F \in B$, and let $d': B \to M$ be an A-derivation into a B-module M. by def of a derivation, $\Rightarrow d'F = \sum_i \frac{\partial F}{\partial T_i} d'T_i$

therefore, d' is entirely determined by the images of T_i.

Let Ω be the free B-module generated by the symbols dT_i (1≤

Let $d: B \to \Omega$ be the map defined by $dF = \sum_i \frac{\partial F}{\partial T_i} dT_i$.

Can check: (Ω, d) satisfies the universal property for $(\Omega^1_{B}$

$\Rightarrow \Omega \cong \Omega^1_{B/A}$. □

Example A.4 Let B be a localization or a quotient of A. Then Ω

Indeed, if $A \twoheadrightarrow B$ surjective, $d(b) = a \underset{=0}{\underline{d(1)}}$ for $a \in A$ an inverse image of

If $B = S^{-1}A$ is a localization of A. For any $b \in B$, there exists a $t \in S$ s.t. $tb \in A \Rightarrow t\,db = d(tb) = 0$, when $db = 0$

since t is invertible in B.

Construction A.5

Let $\rho : B \to C$ be a homo. of A-algebras. Then it follows from the universal property that there exist canonical homo of C-modules

$$\alpha : \Omega^1_{B/A} \otimes_B C \longrightarrow \Omega^1_{C/A}, \quad \beta = \Omega^1_{C/A} \to \Omega^1_{C/B}.$$

by def, $\alpha(db \otimes c) = c\,d(\rho(b))$.

Prop A.6 B = A-algebra

(a) base change. For any A-algebra A', put $B' = B \otimes_A A'$. Then

$$\Omega^1_{B'/A'} \cong \Omega^1_{B/A} \otimes_B B'.$$

($d : B \to \Omega_{B/A}$ induces $d' = d \otimes id_{A'} : B' \to \Omega_{B/A} \otimes_A A'$

$= \Omega^1_{B/A} \otimes_B B'$

Can show $(\Omega^1_{B/A} \otimes_B B', d') \in$ Universal property.)

(b). $B \to C$ homo. of ~~A~~ algebras. α, β as above

then $\Omega^1_{B/A} \otimes_B C \xrightarrow{\alpha} \Omega^1_{C/A} \xrightarrow{\beta} \Omega^1_{C/B} \to 0$

exact

pf $\Leftrightarrow \forall$ N: C-module, the sequence

$$0 \to \mathrm{Hom}_C(\Omega^1_{C/B}, N) \to \mathrm{Hom}_C(\Omega^1_{C/A}, N) \to \mathrm{Hom}_C(\Omega^1_{B/A} \otimes_B C, N)$$

exact. We have $\mathrm{Hom}_C(\Omega_{B/A} \otimes_B C, N) = \mathrm{Hom}_B(\Omega^1_{B/A}, N)$

$0 \to \mathrm{Der}_B(C, N) \to \mathrm{Der}_A(C, N) \to \mathrm{Der}_A(B, N)$ comp with $B \to C$.

by def of a derivation, this seq is exact.

(c) $S \subseteq B$ multiplicative set. Then $S^{-1}\Omega^1_{B/A} \simeq \Omega^1_{S^{-1}B/A}$.

(d). If $C = B/I$, then

$$\begin{tikzcd} I/I^2 \arrow[r, "\delta"] & \Omega^1_{B/A}\otimes_B C \arrow[r, "\alpha"] & \Omega^1_{C/A} \arrow[r] & 0 \\ \bar{b} \arrow[r, mapsto] & db\otimes 1. \end{tikzcd}$$

proof: $I/I^2 = I\otimes_B C$.

$$\begin{tikzcd} \Leftrightarrow \quad 0 \arrow[r] & Der_A(C,N) \arrow[r] & Der_A(B,N) \arrow[r, "\text{restriction}"] & Hom_C(I/I^2,N) \arrow[d, equal] \\ & & & Hom_B(I,N) \end{tikzcd}$$

for any C-

Example A.7 $B = A[T_1,\dots,T_n]$, $F\in B$. $C = B/FB$.

We have $\Omega^1_{C/A} = \bigoplus_{i=1} C\cdot dT_i \Big/ C\cdot dF$ $\quad dF = \sum \frac{\partial F}{\partial T_i} dT_i$.

Lemma A.8 k: field. E/k extension. $K = E[T]/(P(T))$ simple a extens of

(a) If K/E separable, then $\Omega^1_{K/E}=0$, and $\Omega^1_{K/k} \xleftarrow{\sim} \Omega^1_{E/k}\otimes_E K$

(b) If K/E inseparable, then $\Omega^1_{K/E} = K$ and

$$\dim_E \Omega^1_{E/k} \le \dim_K \Omega^1_{K/k} \le \dim_E \Omega^1_{E/k} + 1$$

(c) Suppose K is finite over k. Then K/k is separable iff $\Omega^1_{K/k} = 0$.

proof (1) $P'(T)$ = derivative of $P(T)$

t = image of T in K. Then

$$\Omega^1_{K/E} = KdT/(P'(t))\,dT \cong K/(P'(t)).$$

in case (a), $P'(t) \in K^*$ $\Rightarrow$ $\Omega^1_{K/E} = 0$.

$$\Omega^1_{E/k} \otimes_E K \simeq \Omega^1_{K/k}.$$

in case (b), $P'(t) = 0$, $\Rightarrow$ $\Omega^1_{K/E} \cong K$.

(c). If K/k separable $\Rightarrow$ $\Omega^1_{K/k} = 0$ by (a).

Suppose K/k inseparable, then K is a simple inseparable ext of some subfield E, $\Omega^1_{K/k} \twoheadrightarrow \Omega^1_{K/E} \neq 0$

$\Rightarrow \Omega^1_{K/k} \neq 0$.

2024.05.06

§7 Completions

讨论代数学中的完备化 ← 局部化的简化版本

特别要证明：Completion preserves exactness and Noetherian prop

restricted to f.g. modules

G：topological abelian group, i.e., { (1) $G \in Top$ and $G \in Ab$

(2) $G\times G \xrightarrow{+} G$, $(x,y) \mapsto x+y$; $G \xrightarrow{-} G$, $x \mapsto -x$ are Con

特别：translation $G \xrightarrow{T_a} G$, $x \mapsto x+a$ is

continuous homeomorphism with

Lemma 7.1 G is Hausdorff iff $\{0\}$ is closed in G.

proof ⇒ clear

⇐ If $\{0\} \hookrightarrow G$ is closed, then

= inverse image of $\{0\}$ under "−", which

closed in G

$$\begin{tikzcd}
G \arrow[r, hook, "diag"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & G\times G \arrow[d, "-"] \\
\{0\} \arrow[r, hook] & G
\end{tikzcd}$$

For any $x\in U$, $y \in V$ such that $U\times V \subseteq G\times G \setminus G$, then $U\cap V=$

能选 U,V 使得 $U\cap V = \phi$.

$$\begin{tikzcd}
U\cap V \arrow[r, hook, "closed"] \arrow[d] & U\times V \arrow[d, hook, "open"] \\
G \arrow[r, hook, "closed"] & G\times G
\end{tikzcd}$$

$(x,y) \in U\times V \setminus U\cap V$

不妨设 $x \notin U\cap V$ (否则 $y \notin U$

此时 $x \in U \setminus U\cap V$ ⇐ 两者交

$y \in V$

$T_a: G \longrightarrow G$ 同胚，G在 0 处的拓扑性质 决定了 G在 a 处的拓扑性质

Lemma 7.2 If U is any neigh of $0 \in G$, then $U+a$ is a neigh of

反之, each neigh of $a\in G$ appears in this way.

Lemma 7.3 Let $H = \bigcap_{\substack{0 \in U \\ U \underset{\text{open}}{\subseteq} G}} U$. Then
$$\begin{cases} (1) & H \text{ is a group} \\ (2) & H = \overline{\{0\}}, \text{ closure of } \{0\} \\ (3) & G/H \text{ is Hausdorff} \\ (4) & G \text{ is Hausdorff} \Leftrightarrow H = \{0\}. \end{cases}$$

proof (1) For $x \in H$ and $y \in H$, we show $x+y \in H$.

We need to show: for all open $U \ni 0$, we have $x+y \in U$.

By def of H, we have $x \in U, y \in U$.

Now $G \xrightarrow{T_x} G$, $T_x^{-1}(U) \overset{\text{open}}{\subseteq} G$ and $0 \in \underbrace{T_x^{-1}(U)}_{\text{是0的邻域.}}$

$\Rightarrow x, y \in T_x^{-1}(U)$ since $x, y \in H$

$\Rightarrow T_x(y) = x+y \in U$. ▦

(2) $x \in H \Leftrightarrow x \in U$ for all open $U \ni 0$.

$\Leftrightarrow$ for all $\{0\} \underset{\text{closed}}{\subseteq} V$ $(0 \notin \underset{\text{open}}{G \setminus V})$, we have $x \notin G \setminus V$ (hence $x \in V$)

$\Leftrightarrow x \in \overline{\{0\}} = \bigcap_{\substack{\{0\} \subseteq V \\ \text{closed}}} V$.

(3) By (2), $H \subseteq G$ closed $\Rightarrow \{\bar{0}\}$ in G/H is closed $\Rightarrow$ G/H Hausdorff.

(4) If G is Hausdorff, then $\overline{\{0\}} = \{0\} \Rightarrow H = \{0\}$.

If $H = \{0\}$, then $\{0\}$ is closed $\Rightarrow$ G is Hausdorff. ▦

Definition 7.4 Assume $0 \in G$ has a countable fundamental system of neighborho

Then the completion $\hat{G}$ of G is defined to be the set of equivalence classes of Cauchy sequences.

— A Cauchy sequence in G is defined to be a sequence (x_n) of elements

such that for any neigh $0 \in U$, there is an integer $s(U)$ such that $x_n -$

for all $n, m \geq s(U)$.

— Two cauchy sequence $\{x_n\}$ and $\{y_n\}$ are equivalent if $x_n - y_n \to 0$

— abelian group structure on $\hat{G}$:

$$\text{class}\{x_n\} + \text{class}\{y_n\} := \text{class}\{x_n + y_n\}$$

can check $\hat{G}$ is a top. abelian group

$G \longrightarrow \hat{G}$ dense.

— $\exists$ group homo $G \xrightarrow{\phi} \hat{G}$

$a \longmapsto (a)$ constant seq

ϕ is not injective in general.

Fact $\ker\phi = \bigcap\limits_{\substack{0 \in U \subseteq G \\ \text{open}}} U$

thus by Lemma 7.3, ϕ is injective $\Longleftrightarrow$ G is Hausdorff.

— Functorial property

If $G \xrightarrow{f} H$ continuous homo of abelian top group, then

$f(\text{Cauchy seq}) = \text{Cauchy seq in } H$, thus f induces a homomorphism

$\hat{f} : \hat{G} \longrightarrow \hat{H}$, which is continuous.

For $G \xrightarrow{f} H \xrightarrow{g} K$, we have $\widehat{g \circ f} = \hat{g} \circ \hat{f}$.

Example 7.5 Assume $0 \in G$ has a fundamental system of neighborhoods consisting of subgroups $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n \supseteq \cdots$ and $U \subseteq G$ is a neigh of 0 iff it contains some G_n.

(比如: "p-adic topology" on $\mathbb{Z}$ with $G_n = p^n\mathbb{Z}$)

Claim $G_n \subseteq G$ are both open and closed.

Pf If $g \in G_n$, then $g + G_n \subseteq G_n$ is a neigh of g.

Since $g + G_n \subseteq G_n \Rightarrow G_n$ is open (从而 $\forall g \in G$, $g + G_n$ open)

Hence $G - G_n = \bigcup_{h \notin G_n} (h + G_n)$ is open $\Rightarrow G_n$ is closed. ▦

此时，完备化 $\hat{G}$ 有一个 purely algebraic definition: $\hat{G} \cong \varprojlim G/G_n$.

$$\varprojlim G/G_n \longrightarrow \hat{G} \longrightarrow \varprojlim G/G_n$$ (定义见下面的注)

Given $(\xi_n) \in \varprojlim G/G_n$

Can construct a Cauchy sequence (x_n) by

$x_n =$ any element of G such that $x_n \equiv \xi_n$ in G/G_n

Then $x_{n+1} - x_n \in G_n$

Given any Cauchy sequence (x_n),

($x_{m'} - x_m \in G_n$ for m, m' 充分大

$\Rightarrow \bar{x}_{m'} = \bar{x}_m$ in G/G_n)

Then the image of x_m in G/G_n is ultimately constant, equal to ξ_n (say).

Then
$$\begin{tikzcd} G/G_{n+1} \arrow[r, "\theta_{n+1}"] & G/G_n \\ \xi_{n+1} \arrow[r, mapsto] & \xi_n \end{tikzcd}$$

(ξ_n) coherent sequences.

注 Given a seq in Ab: $\cdots \longrightarrow A_{n+1} \xrightarrow{\theta_{n+1}} A_n \xrightarrow{\theta_n} A_{n-1} \longrightarrow \cdots$

Its inverse limit $= \varprojlim A_n =$ group of coherent sequences (a_n)

(i.e., $a_n \in A_n$ and $\theta_{n+1}(a_{n+1}) = a_n$)

$\varprojlim$ 的左正合性.

Definition 7.6 A seq of inverse systems $0 \longrightarrow \{A_n\} \longrightarrow \{B_n\} \longrightarrow \{C_n\} \longrightarrow 0$ exact iff for all n,

$$\begin{tikzcd}
0 \arrow[r] & A_{n+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & B_{n+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & C_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_n \arrow[r] & B_n \arrow[r] & C_n \arrow[r] & 0
\end{tikzcd}$$

exact

such exact sequence of inverse system induces a sequence

$$0 \longrightarrow \varprojlim A_n \longrightarrow \varprojlim B_n \longrightarrow \varprojlim C_n \longrightarrow 0$$

which is not exact in general. (Needs Mittag-Leffler condition).

Lemma 7.7 If $0 \to \{A_n\} \to \{B_n\} \to \{C_n\} \to 0$ is an exact sequence of inverse systems, then $0 \to \varprojlim A_n \to \varprojlim B_n \longrightarrow \varprojlim C_n$ is always ex

If moreover $\{A_n\}$ is a surjective system ($\forall m$, $A_{m+1} \to A_m$ surj), the

$0 \to \varprojlim A_n \longrightarrow \varprojlim B_n \longrightarrow \varprojlim C_n \longrightarrow 0$ exact)

[⇝ $\varprojlim$: Cat of inverse system $\longrightarrow$ Ab is left exact,

Can define $R^1\varprojlim$. When $\{A_n\}$ surjective, then $R^1\varprojlim\{A_n\} = 0$]

proof $A = \prod_{n=1}^{\infty} A_n \xrightarrow{d^A} A = \prod_{n=1}^{\infty} A_n$

$(a_n) \longmapsto (a_n - \overline{a_{n+1}})$, $\overline{a_{n+1}}$ = image of a_{n+1} under $A_{n+1} \longrightarrow A$

Then $\ker d^A = \varprojlim A_n$ (以后证明 $\operatorname{coker} d^A = R^1\varprojlim A_n$)

Now

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "d^A"'] & B \arrow[r] \arrow[d, "d^B"'] & C \arrow[r] \arrow[d, "d^C"'] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

exact

By Snake Lemma, we get an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \ker d^A \arrow[r] \arrow[d, equal] & \ker d^B \arrow[r] \arrow[d, equal] & \ker d^C \arrow[r] \arrow[d, equal] & \operatorname{coker} d^A \arrow[r] & \operatorname{coker} d^B \arrow[d] \\
& \varprojlim A_n & \varprojlim B_n & \varprojlim C_n & & \operatorname{coker} d^C \arrow[d] \\
& & & & & 0
\end{tikzcd}$$

Now only need to show $\{A_n\}$ surj $\Rightarrow$ d^A surjective ($\operatorname{coker} d^A = 0$).

But this is clear: we only need to solve inductively the equation

$x_n - \overline{x_{n+1}} = a_n$ for $x_n \in A_n$ (for a Given (a_n)).

从 $n=0$ 开始，设 $A_0=0$, ‥‥

<u>Corollary 7.8</u> $0 \to G' \to G \xrightarrow{p} G'' \to 0$ exact seq of ab. groups. Let G with the top defined by a sequence $\{G_n\}$ of subgroups, and give G', G'' the induced topologies, i.e., by the sequence $\{G' \cap G_n\}$, $\{p(G_n)\}$. Then $0 \longrightarrow \hat{G'} \longrightarrow \hat{G} \longrightarrow \hat{G''} \to 0$ exact.

<u>proof</u> Apply 7.7 to $0 \to \{G'/G' \cap G_n\} \longrightarrow \{G/G_n\} \longrightarrow \{G''/p(G_n)\} \to 0$ ∎

Apply 7.8 to $G' = G_n$, then $G'' = G/G_n$ has the discrete topology, so that $\hat{G''} = G''$, hence

<u>Corollary 7.9</u> $\hat{G_n}$ is a subgroup of $\hat{G}$, and $\hat{G}/\hat{G_n} \cong G/G_n$.

In particular, $\hat{\hat{G}} \cong \hat{G}$

$$\begin{tikzcd}
\hat{\hat{G}} \arrow[r, phantom, "\cong"] & \hat{G} \\
\varprojlim \hat{G}/\hat{G_n} \arrow[u, equal, "\hat{G} \text{ with } (\hat{G_n})"'] &
\end{tikzcd}$$

$$\varprojlim \hat{G}/\hat{G_n} \cong \varprojlim G/G_n \cong \hat{G}.$$

Definition 7.10 If $G \xrightarrow{\phi} \hat{G}$ is an isomorphism, then we say G is complete. By Corollary 7.9, $\hat{G}$ is complete.

If G is complete, then $\underset{\underset{\{0\}}{\|}}{\operatorname{Ker}\phi} = \bigcap_{\substack{0 \in U \subseteq G \\ \text{open}}} U = \overline{\{0\}} \Rightarrow$ G Hausdor

Example 7.11 (1) A: ring, $I \subseteq A$ ideal. Take $G = A$, $G_n = I^n$.

The topology on A defined by $\{I^n\}$ is called the I-adic topology.

By 7.1, A is Hausdorff $\Leftrightarrow \bigcap I^n = \{0\}$.

$\phi: A \to \hat{A} = \varprojlim A/I^n$ continuous with $\ker\phi = \bigcap I^n$.

(2) For an A-module M, take $G = M$ and $G_n = I^n M$.

This define the I-adic topology on M, and the completion $\hat{M}$ of M is a topological $\hat{A}$-module (i.e, $\hat{A} \times \hat{M} \to \hat{M}$ is continuous).

If $M \xrightarrow{f} N$ is any A-module homo, then $f(I^n M) = I^n \underset{\subseteq I^n}{f(M)}$

$\Rightarrow$ f is continuous w.r.t I-adic topology.

$\Rightarrow$ f defines $\hat{f}: \hat{M} \to \hat{N}$.

(3) $A = k[x]$, k = fixed, $I = (x)$.

Then $\hat{A} = \varprojlim k[x]/(x^n) = k[\![x]\!]$ is the ring of formal power serie

(4) $A = \mathbb{Z}$, $I = (p)$, $\hat{A} = \mathbb{Z}_p$ = ring of p-adic integers.

$= \left\{ \sum_{n \geq 0} a_n p^n \,\middle|\, 0 \leq a_n \leq p-1 \right\}$ with top $p^n \to 0$ as $n \to$

Definition 7.12 A filtration of M (denoted by (M_n)) is an (infinite) chain:

$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n \supseteq \cdots$ where the M_n are submodules of M.

Say (M_n) is an I-filtration if $IM_n \subseteq M_{n+1}$ for all n.

Say (M_n) is a stable I-filtration if $IM_n = M_{n+1}$ for all sufficiently large n.

example (I^nM) is a stable I-filtration.

Lemma 7.13 (stable I-filtration 定义了相同的拓扑)

若 (M_n) and (M'_n) are stable I-filtrations of M, then they have bounded difference:

$\exists n_0$ s.t. $M_{n+n_0} \subseteq M'_n$ 且 $M'_{n+n_0} \subseteq M_n$ for all $n \geq 0$.

Hence all stable I-filtrations define the same topology on M, namely the I-adic topology.

Pf enough to take $M'_n = I^nM$

Since $IM_n \subseteq M_{n+1} \Rightarrow \underset{\overset{\| \text{def}}{I^nM_0}}{I^nM} \subseteq M_n$ for all n.

also $IM_n = M_{n+1}$ for all $n \geq n_0$ 充分大 $\Rightarrow M_{n+n_0} = I^nM_{n_0} \subseteq I^nM$. ⊞

现介绍处理 filtration 的一种方式: associated graded ring/modules.

Recall 7.14 $A = \bigoplus_{n=0}^{\infty} A_n$, $A_nA_m \subseteq A_{n+m}$ graded ring.

($\Rightarrow A_0$ subring, A_n is an A_0-module)

An element of A_d is called a homogeneous element of degree d.

$I \subseteq A$ an ideal is called a homogeneous ideal if $I = \bigoplus_{d \geq 0} I \cap A_d$, i.e.,

I can be generated by homogeneous elements.

- A homogeneous ideal is a prime iff for any two homogeneous elem
f, g such that $fg \in I$ implies $f \in I$ or $g \in I$.

5月8日

example $S = k[X_0, \cdots, X_n]$ graded with $S_d = \{\Sigma\, a x_0^{i_1} \cdots x_n^{i_d} \mid$ $i_1, \cdots, i$
$\Sigma i_j =$
$a \in$

$$\mathbb{P}^n = \frac{\mathbb{A}^n - \{0\}}{k^\times} = \{(a_0, \cdots, a_n) \in k^{n+1} \setminus 0\} / k^\times$$

$$(a_0, \cdots, a_n) \sim \lambda(a_0, \cdots, a_n), \quad \lambda \in k^\times$$

If $f \in S$ is a polynomial, we cannot use it to define a function

But if f is homogeneous poly of degree d, then

$$f(\lambda a_0, \cdots, \lambda a_n) = \lambda^d f(a_0, \cdots, a_n)$$

So that " f being zero or not " depends only on the equivale
class of $(a_0, \cdots, a_n)$. Thus f gives a function

$$\mathbb{P}^n \longrightarrow \{0, 1\}$$
$$P \longmapsto f(P) = \begin{cases} 0 & \text{if } f(a_0, \cdots, a_n) = 0 \\ 1 & \text{if } f(P) = 1 \end{cases}$$

A : graded ring.

a graded A-module is an A-module M together with a famil
$(M_n)_{n \in \mathbb{Z}}$ of subgroups of M such that $M = \bigoplus_{n \in \mathbb{Z}} M_n$ and $A_m M_n \subseteq M$
$\forall$

$\Rightarrow$ each M_n is an A_0-module.

Elements in M_n are called homogeneous of degree n ~~deg~~ (次数 deg

For $y \in M$, write $^{\exists !}\ y = \sum_n y_n$, $y_n \in M_n$, call y_n the homogeneous component of y.

A homomorphism $f : M \longrightarrow N$ of graded A-modules is an A-module homomorphism such that $f(M_n) \subseteq N_n$ for all n.

If A is a graded ring, let $A_+ = \bigoplus_{n>0} A_n$, then A_+ is an ideal of A

$$\operatorname{Proj} A = \left\{ \mathfrak{p} \,\middle|\, \begin{array}{l} A_+ \not\subseteq \mathfrak{p} \\ \mathfrak{p} \text{ homogeneous prime ideal} \end{array} \right\}$$

$D_+(f) = \{ \mathfrak{p} \in \operatorname{Proj} A \mid f \notin \mathfrak{p} \}$ where $f \in A_+$.

$\uparrow$ principal open subset.

For homogeneous ideal $I \subseteq A$, define $V(I) = \{ \mathfrak{p} \in \operatorname{Proj} A \mid I \subseteq \mathfrak{p} \}$

Lemma 7.15 (1) If I, J homoge ideals $\Rightarrow V(IJ) = V(I) \cup V(J)$.

(2) $V(\sum I_i) = \bigcap V(I_i)$

$\rightsquigarrow$ Zariski topology on Proj A s.t closed subsets are of the form $V(I)$.

For $\mathfrak{p} \in \operatorname{Proj} A$, $T = \left\{ \begin{array}{l} \text{all homogeneous elements} \\ \text{of } A \text{ which are not in } \mathfrak{p} \end{array} \right\}$

Then T is a multiplicative system. Can do localization:

$$A_{(\mathfrak{p})} := T^{-1} A \quad (\text{注意与 } A_{\mathfrak{p}} \text{ 区别})$$

Lemma 7.16 A: graded ring. TFAE:

(1) A is a Noetherian ring.

(2) A_0 is Noetherian and A is f.g. as an A_0-algebra.

proof (1)⇒(2) $A_0 \cong A/A_+ \Rightarrow A_0$ is Noether.

$A_+ \subseteq A$ ideal $\Rightarrow A_+$ is f.g by some $x_1, \dots, x_s$, wh

to be homogeneous of degrees $k_1, \dots$

Let $A' \subseteq A$ be the subring generated by $x_1, \dots, x_s$ over A_0.

We show $A_n \subseteq A'$ for all $n \geq 0$ by induction on n.

- True for $n=0$
- For $n>0$, let $y \in A_n$. Since $y \in A_+ \Rightarrow y = \sum_{i=1}^{s} a_i x_i$, $a_i \in A$

  Since each $k_i > 0 \leadsto$ by induction hypo, each a_i is a poly

  x's with coeff in $A_0 \Rightarrow$ same is true for y

  $\Rightarrow y \in A' \Rightarrow A_n \subseteq A' \Rightarrow A = A'$.

(2)⇒(1) By Hilbert basis thm.

Construction 7.17 A = ring (may not graded), $I \subseteq A$ ideal.

Form a graded ring $A^* = \bigoplus_{n \geq 0} I^n$.

If M is an A-module, (M_n) is an I-filtration of M ($M =$ …, $IM_n \subseteq$ …)

then $M^* = \bigoplus_n M_n$ is a graded A^*-module ($I^m M_n \subseteq M_{m+n}$)

If A is Noetherian, and if $I \subseteq A$ is f.g by $x_1, \dots, x_r$,

$A^* = \bigoplus_{n \geq 0} I^n = A[x_1, \dots, x_r]$ is also Noetherian.

Lemma 7.18 (Stable I-filtration 的判别方法

(定义3个相同的拓扑，之后用它来证 Artin-Rees Lemma)

(I-adic top 的限制还是 I-adic top)

A : Noether ring, M : f.g. A-module. (M_n) : I-filtration of M.

TFAE : (1) M^* is a f.g. A^*-module.

(2) The filtration (M_n) is stable, i.e., $IM_n = M_{n+1}$ for $n \gg 0$.

proof (Key : $M^* = \bigcup M_n^*$, $M_0^* \subseteq \cdots$

M_n^* 由 $M_0 \oplus \cdots \oplus M_n$ 生成)

Each M_n is f.g, hence so is each $Q_n = \bigoplus_{r=0}^{n} M_r$.

Q_n is a subgroup of M^*, but not an A^*-submodule in general.

However, it generate one, namely $M_n^* = M_0 \oplus \cdots \oplus M_n \oplus IM_n \oplus I^2 M_n \oplus \cdots$

Since Q_n is f.g as an A-module $\Rightarrow$ M_n^* is f.g as an A^*-module.

Then M_n^* form an ascending chain, whose union is

$$M^* = \bigcup M_n^*, \quad M_n^* \text{ 由 } M_0 \oplus \cdots \oplus M_n \text{ 生成}.$$

Since A^* is Noether, M^* is f.g as A^*-module.

$\Updownarrow$

the chain stops, i.e., $M^* = M_{n_0}^*$ for some $n_0 > 0$

$\Updownarrow$

$M_{n_0+r} = I^r M_{n_0}$ for some $r \geq 0$

$\Updownarrow$

the filtration is stable. ∎

Prop 7.19 (Artin-Rees Lemma) ← I-adic top 的限制仍是 I-adic top
← stable I-filtration 的限制仍是 stable.

A : Noether ring, $I \subseteq A$ ideal, M : f.g. A-module. (M_n) : stable I-filtration of M.

If $M' \subseteq M$ is a submodule, then $(M' \cap M_n)$ is a stable I-filtration of M'

In particular, If $M_n = I^n M$, then $\exists k \gg 0$, s.t. $I^n M \cap M' = I^{n-k}(I^k M \cap M')$, $\forall n \geq k$

The filtration $I^n M'$ and $I^n M \cap M'$ have bounded difference.

I-adic top on M' coincide with the induced top from I-adic t

proof 利用之前的 stable I-filtration 判别法 (Lemma 7.18).

$$I(M' \cap M_n) \subseteq IM' \cap IM_n \subseteq M' \cap M_{n+1}$$

$\Rightarrow (M' \cap M_n)$ is an I-filtration.

Hence $M^* \underset{\text{submodule}}{\supseteq} \bigoplus_{n \geq 0} M' \cap M_n$ as graded A^*-module.

But M^* is f.g as A^*-module, and A^* is Noether

$\Rightarrow \bigoplus_{n \geq 0} M' \cap M_n$ is also f.g

$\Rightarrow (M' \cap M_n)$ also stable I-filtration by Lemma 7.18. $\blacksquare$

Corollary 7.20 (完备化的正合性)

Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of f.g modu over a Noetherian ring A. Let $I \subseteq A$ be an ideal. Then the sequence of I-adic completions

$$0 \to \widehat{M'} \to \widehat{M} \to \widehat{M''} \to 0 \text{ is exact.}$$

proof Use Corollary 7.8 and Prop 7.19.

对于 localization 有 $M_\mathfrak{p} = M \otimes_A A_\mathfrak{p}$, $A_\mathfrak{p}$ is a flat A-m

但 $\hat{A} \otimes_A M$ 与 $\hat{M}$ 的关系呢？ $\hat{A}$ 是否为 flat A-module？

$$\hat{A}\otimes_A M \longrightarrow \hat{A}\otimes_A \hat{M} \longrightarrow \hat{A}\otimes_{\hat{A}} \hat{M} = \hat{M}$$

不一定是單或滿，但有：

Prop 7.21 For any ring A, if M is f.g, then $\hat{A}\otimes_A M \longrightarrow \hat{M}$ is surjective.

If Moreover A is Noetherian, then $\hat{A}\otimes_A M \to \hat{M}$ is an isom.

proof Completion 正合 $\Rightarrow$ I-adic completion commutes with finite direct sums.

If $F \cong A^n$, then $\hat{A}\otimes_A F \cong \hat{F}$.

Now assume M is f.g so that we have an exact sequence $0 \longrightarrow N \longrightarrow F \longrightarrow M \longrightarrow 0$, where F is free (If A is Noethor, N can also be f.g)

This gives

$$\begin{tikzcd}
\text{If A Noether} \arrow[d] & \hat{A}\otimes_A N \arrow[r] \arrow[d, "\gamma"] & \hat{A}\otimes_A F \arrow[r] \arrow[d, "\beta\cong"] & \hat{A}\otimes_A M \arrow[r] \arrow[d, "\alpha"] & 0 \; \text{正合} \\
0 \arrow[r] & \hat{N} \arrow[r] & \hat{F} \arrow[r, "\delta"] & \hat{M} \arrow[r] & 0 \; \text{正合}
\end{tikzcd}$$

δ surjective, β isom $\Rightarrow$ α surjective (for all f.g. A-module M).

If A is Noether, N is f.g $\Rightarrow$ γ is surjective

diagram chasing $\Rightarrow$ α is injective (or by Snake Lemma)

$\Rightarrow$ α is an isom. ∎

结论总结 A: Noether, $I \subseteq A$ ideal.

(1) $\{\text{f.g. } A\text{-module}\} \longrightarrow \{\hat{A}\text{-module}\}$ exact

$$M \longmapsto \hat{A} \otimes_A M \cong \hat{M}$$

(2) $\hat{A}$ is a flat A-algebra.

(3) $\hat{I} = \hat{A} I \cong \hat{A} \otimes_A I$.

(4) $\widehat{I^n} = (\hat{I})^n$ (因 $\widehat{I^n} = \hat{A} I^n = (\hat{A} I)^n = (\hat{I})^n$)

(5) $I^n / I^{n+1} \cong \hat{I}^n / \hat{I}^{n+1}$ (因 $A/I^n \cong \hat{A}/\hat{I}^n$).

(6) $\hat{I} \subseteq$ Jacob radical of $\hat{A}$ (完备化后落入 Jacob radical)

由(5)知, $\hat{A}$ is complete for the $\hat{I}$-adic topology.

Hence for any $x \in \hat{I}$, $(1-x)^{-1} = 1 + x + x^2 + \ldots$ converges in

$\Rightarrow 1-x$ is a unit $\Rightarrow x \in$ Jacob radical of $\hat{A}$.

Prop 7.22 A Noether local ring, $\mathfrak{m} \subseteq A$ maximal ideal.
Then the $\mathfrak{m}$-adic completion $\hat{A}$ is a local ring with max
ideal $\hat{\mathfrak{m}}$.

proof $A/\mathfrak{m} \cong \hat{A}/\hat{\mathfrak{m}}$ field $\Rightarrow \hat{\mathfrak{m}}$ is a maximal ideal of $\hat{A}$.

But $\hat{\mathfrak{m}} \subseteq$ Jacob radical of $\hat{A} \Rightarrow \hat{\mathfrak{m}} =$ Jacob radical of
$\subseteq \hat{\mathfrak{m}}$ and $\hat{\mathfrak{m}}$ is the unique maximal ideal

$\Rightarrow \hat{A}$ is local.

A: Noether

$\mathfrak{p} \in \operatorname{Spec} A \Rightarrow A_{\mathfrak{p}}$ local ring. $\rightsquigarrow$ completion $\widehat{A_{\mathfrak{p}}}$ of $A_{\mathfrak{p}}$ w.r.t $\mathfrak{p}A_{\mathfrak{p}}$

is still a local ring

Krull's thm tells you how much information we lose after completion.

Prop 7.23 (Krull's thm) A: Noether ring, $I \subseteq A$ ideal, M: f.g. A-module.

$\hat{M}$: I-adic completion of M.

then $$\operatorname{Ker}(M \longrightarrow \hat{M}) \underset{\text{之前已证}}{=} \bigcap_{n=1}^{\infty} I^n M = \left\{ x \in M \,\middle|\, \begin{array}{l} \exists y \in I \text{ s.t} \\ (1+y)x = 0 \end{array} \right\}$$

特别：若 A Noether domain, and $I \neq (1)$, then $\bigcap I^n = (0)$.

proof Let $$E = \operatorname{Ker}(M \longrightarrow \hat{M}) = \bigcap_{\substack{0 \in U \\ U \subseteq M \\ \text{open}}} U$$

$\Rightarrow$ the top induced on E is trivial (E is the only neigh of $0 \in E$).

But the induced top on E coincide with its I-adic topo,

since IE is a neigh in the I-top $\Rightarrow IE = E$.

Since M is f.g and A Noether $\Rightarrow E$ is also f.g.

$\Rightarrow \exists \alpha \in I$ s.t $(1-\alpha)E = 0$

the converse is obvious if $(1-\alpha)x = 0$, then $x = \alpha x = \alpha^2 x$

$= \cdots \in \bigcap_{n=1}^{\infty} I^n M = E$

∎

Remark 7.24 (不记#)

(1) If $S = 1 + I$, then 7.23 $\Rightarrow \mathrm{Ker}(A \to \hat{A}) = \ker(A \to S^{-1}$

For any $\alpha \in \hat{I}$, $(1-\alpha)^{-1} = 1 + \alpha + \alpha^2 + \cdots$ converges in

$\Rightarrow$ element of S becomes a unit in $\hat{A}$.

By the universal prop of $S^{-1}A \Rightarrow \exists$ natural homo $S^{-1}A \to \hat{A}$

且 $S^{-1}A \to \hat{A}$ injective.

Thus $S^{-1}A$ is a subring of $\hat{A}$.

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & S^{-1}A \arrow[d, hook] \\ & \hat{A} \end{tikzcd}$$

且 $\mathrm{Ker}(A \to S^{-1}A) = \ker(A \to$

(2) If A Non-Noether, then Krull thm may false.

$A = C^{\infty}(\mathbb{R}, \mathbb{R}) \supseteq I = \{f \in A \mid f(0) = 0\}$

I is maximal with $A/I \cong \mathbb{R}$.

I is generated by the identity function x, and $\bigcap_{n=1}^{\infty} I^n = \{f \in A$

f annihilated by some $1 + \alpha$ $(\alpha \in I) \Leftrightarrow f$ vanishes identically in s

neigh. of 0.

e^{-1/x^2} which is not identically zero near 0, but has vanishing

derivative at 0, thus $e^{-1/x^2} \in \ker(A \to \hat{A}) \neq \ker(A \to S$

Thus A is Not Noether.)

Corollary 7.25 A Noether, $I \subseteq A$ ideal s.t. $I \subseteq$ Jacob radical of A
M: f.g. A-module

Then the I-adic top on M is Hausdorff, i.e., $\bigcap I^n M = 0$.

proof In this case, every element of $1+I$ is a unit. ∎

Corollary 7.26 A: Noether local ring with maximal ideal $\mathfrak{m}$.
M: f.g. A-module

then the $\mathfrak{m}$-adic top on M is Hausdorff.

In particular, the $\mathfrak{m}$-adic top on A is Hausdorff. ∎

[illegible] A: Noether ring. then A的完备化仍是Noether,
特别 $A[[X_1, \dots, X_n]]$ is Noether.

7.27 Associated graded ring

之前是 $\bigoplus_{n=0}^{\infty} I^n$

A: ring, $I \subseteq A$ ideal. Define $G(A) = G_I(A) = \bigoplus_{n=0}^{\infty} I^n / I^{n+1}$.

$G(A)$ is a graded ring: for $x_n \in I^n$, denote $\overline{x_n}$ = image of x_n in I^n/I^{n+1}

define $\overline{x_m} \cdot \overline{x_n} := \overline{x_m x_n}$ in I^{m+n}/I^{m+n+1}

与代表元选取无关.

Similarly, for an A-module M, and (M_n): I-filtration of M, we define $G_n(M) = M_n/M_{n+1}$, $G(M) = \bigoplus_{n \geq 0} M_n/M_{n+1}$, which is a graded $G(A)$-module.

Prop 7.26 A: Noether ring. $I \subseteq A$ ideal. Then

(1) $G_I(A)$ is Noether.

(2) $G_I(A)$ and $G_{\hat{I}}(\hat{A})$ are isomorphic as graded rings

(3) If M is a f.g. A-module, and (M_n) is a stable I-filtration, then $G(M)$ is a f.g graded $G_I(A)$-module.

proof (1) A: Noether $\Rightarrow$ I is f.g. by some $x_1, \cdots, x_n$.

Let $\bar{x}_i$ = image of x_i in I/I^2. Then $G(A) = A/I[\bar{x}_1, \cdots, \bar{x}_n]$

A/I Noether $\Rightarrow G(A)$ Noether.

(2) $I^n/I^{n+1} \cong \hat{I}^n/\hat{I}^{n+1}$.

(3) $\exists n_0$ s.t $M_{n_0+r} = I^r M_{n_0}$, $\forall r \geq 0$.

$\Rightarrow G(M)$ is generated by $\bigoplus_{0 \leq n \leq n_0} G_n(M)$

each $G_n(M) = M_n/M_{n+1}$ is Noether and annihilate I, hence is a f.g. A/I-module

$\Rightarrow \bigoplus_{0 \leq n \leq n_0} G_n(M)$ is generated by a finite number elements (as A/I-module)

$\Rightarrow G(M)$ is f.g as a $G(A)$-module.

Lemma 7.27 (不证) $A \xrightarrow{\phi} B$ homo of filtered groups, i.e., $\phi(A_n) \subseteq B_n$, and let $G(\phi): G(A) \to G(B)$, $\hat{\phi}: \hat{A} \to \hat{B}$ the induced homo.

Then ① $G(\phi)$ injective $\Rightarrow$ $\hat{\phi}$ injective

② $G(\phi)$ surjective $\Rightarrow$ $\hat{\phi}$ surjective.

Pf

$$\begin{tikzcd}
0 \arrow[r] & A_n/A_{n+1} \arrow[r] \arrow[d, "G_n(\phi)"] & A/A_{n+1} \arrow[r] \arrow[d, "\alpha_{n+1}"] & A/A_n \arrow[r] \arrow[d, "\alpha_n"] & 0 \\
0 \arrow[r] & B_n/B_{n+1} \arrow[r] & B/B_{n+1} \arrow[r] & B/B_n \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow$ $0 \to \ker G_n(\phi) \to \ker \alpha_{n+1} \to \ker \alpha_n \to \operatorname{coker} G_n(\phi) \to \operatorname{coker} \alpha_{n+1} \to \operatorname{coker} \alpha_n \to 0$

From this, we see by induction on n that $\ker \alpha_n = 0$ (in ①), or $\operatorname{coker} \alpha_n = 0$ (in ②)

① α_0 injective. $G_n(\phi)$ inj, α_n inj $\Rightarrow$ α_{n+1} injective

② We have $\ker \alpha_{n+1} \to \ker \alpha_n$ surj. Take inverse limit of α_n, and apply "left exact of $\varprojlim$" $\Rightarrow$ $\operatorname{Coker} G_n(\phi) = 0$ and $\hat{\phi}$ surjective ☒

(利用 G(M) 的性质 可推 M 的性质)

7.28 A: ring, I: ideal, M: A-module. (M_n): I-filtration of M.

Suppose that A is complete for the I-adic top and that M is Hausdorff in its filtration top (i.e., $\bigcap_n M_n = 0$).

(1) Suppose that $G(M)$ is a f.g. $G(A)$-module, then M is a f.g. A-mod.

(2) If $G(M)$ is a Noether $G(A)$-module, then M is a Noether A-module.

(1) Pick a finite set of generators of $G(M)$, and split them into their homogeneous component, say $\xi_i = \bar{x}_i$, $x_i \in M_{n(i)}$.

$F^i = A$ with stable I-filtration $F^i_k = I^{k+n(i)}$ and put

$F = \bigoplus_{i=1}^{r} F^i$.

Now $F^i \xrightarrow{\phi} M$ is a homo of filtered groups

$1 \longmapsto x_i$

$G(F^i) \xrightarrow{G(\phi)} G(M)$ ——— of $G(A)$-module

$\overset{\text{define}}{\rightsquigarrow}$ $\begin{cases} F \xrightarrow{\phi} M \\ G(F) \xrightarrow{G(\phi)} G(M). \end{cases}$

By construction $G(\phi)$ surj $\Rightarrow$ $\hat{\phi}$ surjective.

Consider

$$\begin{tikzcd}
F \arrow[r, "\phi"] \arrow[d, "\alpha"'] & M \arrow[d, "\beta"] \\
\hat{F} \arrow[r, "\hat{\phi}"] & \hat{M}
\end{tikzcd}$$

F free, $\hat{A} = A$ complete $\Rightarrow \alpha$ is an isom.

M Hausdorff $\Rightarrow \beta$ injective. But $\hat{\phi}$ surj $\Rightarrow \phi$ surjective

this means that $x_1, \dots, x_r$ generate M as an A-module

(2) Need to show every submodule $M' \subseteq M$ is f.g.

Let $M'_n = M' \cap M_n$. Then (M'_n) is an I-filtration of M'

$M' \to M_n$ gives an injective homo $M'_n / M'_{n+1} \longrightarrow M_n / M_{n+1}$

$\Rightarrow G(M') \hookrightarrow G(M) \underset{\text{Noether}}{\rightarrow} G(M')$ is f.g.

Since $\bigcap M'_n \subseteq \bigcap M_n = 0 \Rightarrow M'$ Hausdorff. By (1) $\Rightarrow M'$ is f.g. ▦

Theorem 7.29 (Noether环的完备化仍是Noether环)

If A is Noether and if $I \subseteq A$ is an ideal, then $\hat{A}$ is Noether.

pf $G_I(A) = G_{\hat{I}}(\hat{A})$ is Noether.

By 7.28 $\Rightarrow \hat{A}$ is Noether (note that $\hat{A}$ complete and Hausdorff) □

Corollary 7.30 If A is a Noether ring, the power series ring $B = A[[X_1, \dots, X_n]]$ is Noetherian. In particular, $k[[X_1, \dots, X_n]]$ is Noether for any field k.

pf $A[X_1, \dots, X_n]$ Noether, and $A[[X_1, \dots, X_n]]$ is the completion of $A[X_1, \dots, X_n]$ for the $(X_1, \dots, X_n)$-adic topology.

Then apply 7.29. ▨

2024年5月15日

§8 Dimension Theory

$A \neq 0$ ring. $\mathfrak{p} \in \operatorname{Spec} A$.

(1) <u>height of $\mathfrak{p}$</u> $= ht(\mathfrak{p}) = \sup\left\{ n \,\middle|\, \begin{array}{l} \exists \text{ prime chain of length } n \text{ sta} \\ \mathfrak{p}_0 = \mathfrak{p} \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n \end{array} \right.$

$= \dim A_{\mathfrak{p}}$

$\leftarrow$ Krull dim of $\mathfrak{p}$.

$ht(\mathfrak{p}) = 0 \Leftrightarrow \mathfrak{p}$ is a minimal prime ideal of A.

(2) For any ideal $I \subsetneq A$, <u>height of I</u> $= ht(I) = \inf\{ ht(\mathfrak{p}) \mid I \subseteq \mathfrak{p} \}$

(3) $\dim A = \sup\{ ht(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec} A \} =$ Krull dim of A

$\dim A < \infty$ 時, $\dim A =$ length of the longest prime chai

$\dim\begin{pmatrix}\text{principal} \\ \text{ideal domain}\end{pmatrix} = 1$

By definition, $ht(\mathfrak{p}) = \dim A_{\mathfrak{p}}$ for $\mathfrak{p} \in \operatorname{Spec} A$

For any $\underset{\text{ideal}}{I} \subsetneq A$, $\underset{I之上}{\dim A/I} + \underset{I之下}{ht(I)} \leq \dim A$

(4) $M \neq 0$ A-module.

dimension of $M = \dim M = \dim(A/\operatorname{ann}(M))$ [$M=0$時

If A is Noether, and if $M \neq 0$ is finite over A, the fo

conditions are equivalent:

clear $\rightarrow$ (a) M is an A-module of finite length

(b) $A/\operatorname{ann}(M)$ is Artinian

已证 $<$ (c) $\dim M = 0$

(a) $\Rightarrow$ (c) (hence (b)) Suppose $\ell(M)<\infty$. Replace A by $A/\mathrm{ann}(M)$, we may assume $\mathrm{ann}(M)=0$.

若 $\dim A>0$, take a minimal prime $\mathfrak{p}\in\mathrm{Spec}\,A$ which is not maximal.

Since M is f.g over A and $\mathrm{ann}(M)=0 \Rightarrow M_{\mathfrak{p}}\neq 0$.

$\Rightarrow \mathfrak{p}$ is a minimal element of $\mathrm{Supp}(M)$

$\Rightarrow \mathfrak{p}\in \mathrm{Ass}(M)$. Thus M contains a submodule isomorphic to $A/\mathfrak{p}$.

Since $\dim A/\mathfrak{p}>0$, we have $\ell(A/\mathfrak{p})=\infty$, contradiction (与 $\ell(M)<\infty$ 矛盾).

Therefore $\dim A\ (=\dim M)=0$. $\square$

将来在研究: Dimension theory of Noether local ring $(A,\mathfrak{m})$.

$$\dim A = \text{degree of Hilbert polynomial } \chi^{A}_{\mathfrak{m}}(n)$$

$$= \mathrm{length}(A/\mathfrak{m}^n) \quad n\gg 0.$$

参: Serre, local algebra.

Review on Integer-valued polynomials.

binomial 二项式 $Q_k(X)=\binom{X}{k}=\dfrac{X(X-1)\cdots(X-k+1)}{k!}$, $k\in\mathbb{N}$

$$Q_0(X)=1,\quad Q_1(X)=X.$$

difference operator $\Delta f(n)=f(n+1)-f(n)$.

One has $\Delta Q_k=Q_{k-1}$ for $k>0$.

Lemma 8.1 For $f\in\mathbb{Q}[X]$, 以下等价:

(1) f is a $\mathbb{Z}$-linear combination of the binomial poly Q_k.

(2) $f(n)\in\mathbb{Z}$, $\forall n\in\mathbb{Z}$

(3) $f(n)\in\mathbb{Z}$, $\forall n\gg 0$

(4) Δf has property (1), and there is at least one integer n such tha

proof We prove $(4)\Rightarrow(1)$

$\Delta f=\sum e_k Q_k$, $e_k\in\mathbb{Z}$

$\Rightarrow f=\sum e_k Q_{k+1}+e_0$, $e_0\in\mathbb{Q}$

But f takes at least one integral value on $\mathbb{Z}$

$\Rightarrow e_0\in\mathbb{Z}\Rightarrow(1)$

$(3)\Rightarrow(4)\Leftrightarrow(1)$

prove by induction on $\deg(f)$.

Then by induction hypothesis, Δf has property (1).

hence (4) true for f by $((4)\Leftrightarrow(1))$.

Definition 8.2 A polynomial f having properties (1)-(4) above is calle an integer-valued polynomial.

If f is such a polynomial, we write $e_k(f)$ for the coefficients of in the decomposition of f:

$$f=\sum e_k Q_k.$$

One has $e_k(f)=e_{k-1}(\Delta f)$ if $k>0$.

$\Rightarrow$ If $\deg f\le k$, $e_k(f)$ is equal to the constant polynomial $\Delta^k(f)$.

we have $f(x)=e_k(f)\frac{x^k}{k!}+g(x)$ with $\deg g<k$.

If $\deg f = k$, one has $f(n) \sim e_k(f)\frac{n^k}{k!}$ for $n \to \infty$.

Hence $e_k(f) > 0 \iff f(n) > 0$ for all large enough n.

We say $f: \mathbb{Z} \longrightarrow \mathbb{Z}$ is polynomial like if $\exists$ polynomial $P_f(X)$ such that $f(n) = P_f(n) \ \forall n \gg 0$

Lemma 8.3 TFAE:

(1) f is polynomial like.

(2) Δf is polynomial like.

(3) $\exists r \geq 0$ s.t $\Delta^r f(n) = 0 \quad \forall n \gg 0$.

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) clear.

(2) $\Rightarrow$ (1) $P_{\Delta f} = \Delta f$, $P_{\Delta f}$ integer valued

$\Rightarrow \exists$ integral valued polynomial R s.t $\Delta R = P_{\Delta f}$.

The function $g(n) = f(n) - R(n)$ satisfies $\Delta g(n) = 0 \ \forall n \gg 0$

$\Rightarrow g(n) \equiv e_0 \quad \forall n \gg 0$

$\Rightarrow f(n) = R(n) + e_0 \quad \forall n \gg 0 \Rightarrow f$ is polynomial like.

(3) $\Rightarrow$ (1) Follows from (2) $\Rightarrow$ (1) applied r-times. ∎

也可参考生成函数 $\sum_{n=0}^{\infty} f(n) t^n \in \mathbb{Z}[[t]]$

8.4 Poincaré Series of graded modules.

$A = \bigoplus_{n=0}^{\infty} A_n$ Noether graded ring ($\Rightarrow A_0$ Noether and A is generated as A_0-algebra by some homog elements $x_i \in A_{k_i}$ ($k_i > 0$), $1 \leq i \leq s$)

重要.

$M=\bigoplus M_n$ f.g. graded A-module.

Then M is generated by a finite number of homogeneous elemen

$m_j \in M_{r_j}$ $(1\leq j\leq t)$, each M_n is f.g as an A_0-module.

$$\lambda: \left\{\begin{matrix}\text{f.g} \\ A_0\text{-module}\end{matrix}\right\} \longrightarrow \mathbb{Z} \quad \text{any additive function}$$

(比如 A_0=Artin环时, $\lambda(M)=\text{length}_{A_0}(M)$ / 也可除 掉 $\text{length}_{A_0}A_0$)

Poincare series of M is the generating function of $\lambda(M_n)$, i.e.,

$$P(M,t)=\sum_{n=0}^{\infty}\lambda(M_n)t^n \quad \text{in } \mathbb{Z}[\![t]\!].$$

<u>Thm 8.5 [Hilbert, Serre]</u> $P(M,t)$ is a rational function in t of the for

$$\frac{f(t)}{\prod_{i=1}^{s}(1-t^{k_i})}, \quad f(t)\in\mathbb{Z}[t]$$

A is generated as an A_0-

by $x_i \in A_{k_i}$, $k_i>0$, $1\leq i$

We denote $d(M)$ = order of the pole of $P(M,t)$ at $t=1$

($d(A)$ same.) 之后要与 dim A 建立联系.

$d(M)$ measure the "size" of M relative to λ.

<u>Pf</u> Induction on s = number of generators of A over A

If $s=0 \Rightarrow A_n=0$ $(\forall n>0)$

$A=A_0$ and M is a f.g. A_0-module

$\Rightarrow M_n=0$ for $n\gg 0 \Rightarrow P(M,t)$ is a polynomial $\Rightarrow d($

Suppose $s>0$ and thm true for $s-1$.

From $M_n \xrightarrow{x_s} M_{n+k_s}$, get $0 \to K_n \to M_n \xrightarrow{x_s} M_{n+k_s} \to L_{n+k_s} \to 0$ exact

Put $M \supseteq K = \oplus K_n$, $L = \oplus L_n$ (Quotient of M).

K and L are f.g. A-module and annihilated by x_s

$\Rightarrow$ K and L are $A_0[x_1, \ldots, x_{s-1}]$-module (由归纳假设, $P(K,t)$ and $P(L,t)$ 是有理多项式)

$$\lambda(K_n) - \lambda(M_n) + \lambda(M_{n+k_s}) - \lambda(L_{n+k_s}) = 0$$

multiply by t^{n+k_s} and sum w.r.t n, we get:

$$(1-t^{k_s})P(M,t) = P(L,t) - t^{k_s}P(K,t) + g(t)$$, $g(t)$ is a polynomial

By induction hypothesis $\Rightarrow$ result

$$\Rightarrow \sum_n \lambda(K_n)t^{n+k_s} - \sum_n \lambda(M_n)t^{n+k_s} + \sum_n \lambda(M_{n+k_s})t^{n+k_s} - \sum \lambda(L_{n+k_s})t^{n+k_s} = 0$$

$$\parallel$$

$$P(K,t)t^{k_s} - P(M,t)t^{k_s} + (P(M,t) - \text{多项式}) - (P(L,t) - \text{多项式}) = 0.$$ ☒

Corollary 8.6 If each $k_i = 1$ (比如 A 作为 A_0 代数由 A_1 中元素生成), 此时 $P(M,t) = \frac{f(t)}{(1-t)^s}$. Then for all $n \gg 0$, $\lambda(M_n)$ is a polynomial in n (with rational coefficients) of degree $d(M)-1$.

(取整值的多项式的系数不一定为 $\mathbb{Z}$, 如 $\frac{1}{2}x(x+1)$).

注意: 对于 $n \gg 0$, $\lambda(M_0) + \cdots + \lambda(M_n)$ poly of degree $d(M)$.

上述多项式f 称为 $M=\bigoplus_{n=0}^{\infty} M_n$ 的 Hilbert function/Polynomial.

(将它用到 $(M,\{I^nM\})$, $\bigoplus I^nM/I^{n+1}M$ $\rightsquigarrow$ length $M/I^nM = \sum_{i=0}^{n-1}$ leng

Proof, $\lambda(M_n) =$ coeff of t^n in $\frac{f(t)}{(1-t)^s}$, f(t)中也可能有 $1-t$ 因子

约分, 消去公因子 $(1-t)^{\bullet}$, may assume $s=d=d(M)$ and f(

Suppose $f(t)=\sum_{k=0}^{N} a_k t^k$ with $f(1)=\sum a_k \neq 0$.

Since $\frac{1}{(1-t)^d} = (1+t+t^2+\cdots)^d = \sum_{k=0}^{\infty}\binom{d+k-1}{d-1}t^k$

We have $\lambda(M_n) =$ coeff of t^n in $\frac{f(t)}{(1-t)^d} = f(t)\sum_{k=0}^{\infty}\Big($

$\lambda(M_n) =$ coeff of t^n in $\sum_{k'=0}^{N} a_{k'} t^{k'} \sum_{k=0}^{\infty}\binom{d+k-1}{d-1}t^k$

$$\lambda(M_n) = \sum_{k=0}^{N} a_k \cdot \binom{d+n-k-1}{d-1} \text{ for all } n \geq N$$

(此时 $n-k\geq 0$, $\forall k=$

which is a polynomial in n, with leading term

$$\left(\sum a_k\right)\frac{n^{d-1}}{(d-1)!} \neq 0$$

of degree $d-1$.

Proposition 8.7 If $x\in A_k$ is not a zero-divisor in M (i.e., $xm=0 \Rightarrow$

then $d(M/xM) = d(M)-1$.

proof $x: M_n \longrightarrow M_{n+k}$

get exact sequence $0 \to K_n \to M_n \xrightarrow{x} M_{n+k} \longrightarrow L_{n+k} \longrightarrow 0$

$$L_{n+k} \parallel M_{n+k}/xM_n$$

同8.5证明, get $P(M/xM, t) = P(M,t) - t^k P(M,t) + g(t)$ ← polynomial

$$= (1-t^k)\, P(M,t) + g(t)$$

← 关于 $t=1$ 处的 pole 的 order 减少 1

$\Rightarrow d(M/xM) = d(M) - 1$. ▦

Example 8.8 A_0: Artin ring. $A = A_0[X_1, \ldots, X_s]$.

A_n is a free A_0-module generated by $\{x_1^{m_1} \cdots x_s^{m_s} \mid \sum m_i = n\}$

A_n are free of rank $\binom{n+s-1}{s-1}$.

Choose $\lambda = \dfrac{\ell(\ )}{\ell(A_0)}$ length function.

Then $P(A,t) = \lambda(A_0) + \lambda(A_1)t + \cdots$

$$= \frac{1}{(1-t)^s}$$ (与 坝 的展开对比)

▦

Thus $d(A) = s$.

Proposition 8.9 A: Noether local ring with maximal ideal $\mathfrak{m}$.

$\underline{q} \subseteq A$ ideal s.t $\sqrt{\underline{q}} = \mathfrak{m}$ [$\mathfrak{m}$-primary ideal].

M: f.g. A-module. (M_n): stable $\underline{q}$-filtration of M.

Then (1). M/M_n is of finite length for ~~$n\geq 0$~~ $n \geq 0$.

(2) For $n \gg 0$, $n \longmapsto$ length M/M_n is a polynomial $g(n)$ of degree $\leq$

where s is the least number of generators of $\underline{q}$.

(3) The degree and the leading coefficient of $g(n)$ depend only

M and $\underline{q}$, not on the filtration (与 $(\underline{q}^n M)$ 给出的[illegible]

proof (1) We show M/M_n is of finite length for $n \gg 0$.

$G(A) = \bigoplus_n \underline{q}^n/\underline{q}^{n+1}$, $G_0(A) = A/\underline{q}$ Artin Local ring.

$G(A)$ Noether and $G(M) = \bigoplus M_n/M_{n+1}$ is f.g. $G(A)$-mod

each $G_n(M) = M_n/M_{n+1}$ is a Noether A-module, annihilat- by $\underline{q}$

$\Rightarrow G_n(M)$ is a Noether $\underset{\overset{\uparrow}{\text{Artin}}}{A/\underline{q}}$-module $\Rightarrow G_n(M)$ is of finite length

$\Rightarrow M/M_n$ is of finite length and

$$l(M/M_n) = \sum_{r=1}^{n} l(M_{r-1}/M_r).$$

② If $x_1, \dots, x_s$ generate q, the image $\bar{x}_i \in q/q^2$ generate $G(A)$ as an A/q-algebra, and each $\bar{x}_i$ has degree 1.

By 8.6 ⇒ $\ell(M_n/M_{n-1}) = f(n)$, $n \gg 0$, where $f(n)$ is a polynomial of degree $\leq s-1$ $(n \gg 0)$.

From $\ell(M/M_n) = \sum_{r=1}^{\infty} \ell(M_{r-1}/M_r)$, we have

$$f(n) = \ell(M/M_{n+1}) - \ell(M/M_n)$$

$\Rightarrow \forall\, n \gg 0$, $\ell(M/M_n)$ is a polynomial $g(n)$ of degree $\leq s$.

③ Let $(\tilde{M}_n)$ be another stable q-filtration of M, and let $\tilde{g}(n) = \ell(M/\tilde{M}_n)$. The two filtrations have bounded differences, i.e. $\exists n_0$ s.t. $M_{n+n_0} \subseteq \tilde{M}_n$

$\tilde{M}_{n+n_0} \subseteq M_n$ for $n > 0$

$\Rightarrow g(n+n_0) \geq \tilde{g}(n)$, $\tilde{g}(n+n_0) \geq g(n)$

Since g and $\tilde{g}$ are polynomials for all large n

$\Rightarrow \lim_{n\to\infty} \frac{g(n)}{\tilde{g}(n)} = 1 \Rightarrow g$ and $\tilde{g}$ have the same degree and leading coeff. ▦

5月20日

Definition 8.10 在8.9的条件下，定义多项式 $\chi_q^M(n)$ s.t

$$\chi_q^M(n) = \ell(M/q^n M) \quad \forall n \gg 0$$

$\chi_q(n) := \chi_q^A(n)$ is called the characteristic polynomial of the ideal q satisfying $\sqrt{q} = m$.

Corollary 8.11 $\deg \chi_q(n) = \deg \chi_m(n)$, 从而 $\deg \chi_q$ 与 q 的选

Pf $m \supseteq q \supseteq m^r \Rightarrow m^n \supseteq q^n \supseteq m^{rn}$

$\Rightarrow \chi_m(n) \le \chi_q(n) \le \chi_m(rn)$ for all $n \gg 0$.

let $n \to \infty$, and $\chi_\cdot(n)$ are polynomial in n $\Rightarrow$ resu

Definition 8.12 We define $d(A) := \deg \chi_q(n) = \deg \chi_m(n)$.

Note $n \gg 0$, $n \mapsto \text{length } A/m^n = \text{length } A/m + \text{length } m/m^2$
$+ \cdots + \text{length } m^{n-}$

is of degree $d(G_m(A))$, $G_m(A) = \bigoplus_{i=0}^{\infty} m^i/m^{i+1}$

↑

order of pole at 1

8.13 Dimension theory of Noether local rings.

A: Noether local ring with maximal ideal $\mathfrak{m}$.

$\mathfrak{q} \subseteq A$ with $\sqrt{\mathfrak{q}} = \mathfrak{m}$ ($\mathfrak{q}$ is $\mathfrak{m}$-primary ideal)

$\delta(A)$ = least number of generators of an $\mathfrak{m}$-primary ideal of A

(会证与 $\mathfrak{m}$-primary ideal 选取无关)

目标：$\delta(A) = d(A) = \dim(A)$.

我们证明：$\delta(A) \geqslant d(A) \geqslant \dim A \geqslant \delta(A)$.

Prop 8.9 表明 $\delta(A) \geqslant d(A)$

Prop 8.14 $(A, \mathfrak{m}, \mathfrak{q})$ as before. M: f.g. A-module. $x \in A$ non-zero divisor in M.

Then $\deg \chi_{\mathfrak{q}}^{M/xM} \leq \deg \chi_{\mathfrak{q}}^{M} - 1$

(特例：If $x \in A$ is not a zero divisor in A, then $d(A/x) \leq d(A) - 1$)

~~proof~~ $N = xM \cong M$ as A-module.

$M' = M/xM$

Let $N_n = N \cap \mathfrak{q}^n M$. Then $0 \to N/N_n \to M/\mathfrak{q}^n M \to M'/\mathfrak{q}^n M' \to 0$

If $g(n) = \ell(N/N_n)$, we have $g(n) - \chi_{\mathfrak{q}}^{M}(n) + \chi_{\mathfrak{q}}^{M'}(n) = 0 \quad \forall n \gg 0$.

By Artin-Rees, (N_n) is a stable $\mathfrak{q}$-filtration of N.

Since $N \cong M$, $g(n)$ and $\chi_{\mathfrak{q}}^{M}(n)$ have the same leading term

$\Rightarrow \deg \chi_{\mathfrak{q}}^{M'}(n) \leq \deg \chi_{\mathfrak{q}}^{M}(n) - 1$. 田

Prop 8.15 $d(A) \geq \dim(A)$ [特别：对Noether local ring A, $\dim A$ is finite]

proof Induction on $d = d(A)$.

If $d=0$, then $\ell(A/\mathfrak{m}^n)$ is constant for all large n, we have $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some $n \Rightarrow \mathfrak{m}^n = 0$ by Nakayama $\Rightarrow A$ is Artin and $\dim A = 0$.

Suppose $d>0$ and let $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ be any chain of prime ideals in A

Let $x \in \mathfrak{p}_1 - \mathfrak{p}_0$. Let $A' = A/\mathfrak{p}_0$ and $x' = \bar{x} \in A' = A/\mathfrak{p}_0$.

Then $x' \neq 0$ and A' is an integral domain $\Rightarrow d(A'/x') \leq d(A') - 1$.

Also, if $\mathfrak{m}'$ is the maximal ideal of A', $A'/\mathfrak{m}'^n$ is a homomorphic imag of $A/\mathfrak{m}^n \Rightarrow \ell(A/\mathfrak{m}^n) \geq \ell(A'/\mathfrak{m}'^n)$.

$$\Rightarrow d(A) \geq d(A') \geq d(A'/x') + 1 \quad (d(A'/x') \leq d(A) - 1 = d-1)$$

By induction hypothesis $\Rightarrow$ the length of any chain of prime ideals in A'/x' is $\leq d-1$

But the images of $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ in A'/x' form a chain of len $r-1$. $\Rightarrow r-1 \leq d-1 \Rightarrow r \leq d \Rightarrow \dim A \leq d$.

Remark 8.16 height $\mathfrak{p} = \dim A_{\mathfrak{p}}$.

In a noetherian ring, every prime ideal has finite height, and therefore the set of prime ideals in a Noether ring satisfies the descending chain conditions. (有趣)

Proposition 8.17 Let A be a Noether local ring of dim d. Then there exists an $\mathfrak{m}$-primary ideal in A generated by d-elements $x_1, \dots, x_d$ and therefore $\dim A \geqslant \delta(A)$. (结合 8.15 + 8.9 $\Rightarrow \delta(A) \geq d(A) \geq \dim A \geq \delta(A)$)

总结: $\dim A = \deg \chi_{\mathfrak{m}}(n) = \underset{n \gg 0}{\ell(A/\mathfrak{m}^n)} = \delta(A) =$ least number of generators of an $\mathfrak{m}$-primary ideal of A.

proof construct $x_1, \dots, x_d$ inductively in such a way that every prime ideal containg $(x_1, \dots, x_i)$ has height $\geq i$ for each i.

$\Rightarrow$ 结论. 事实上, if $\mathfrak{p}$ is a prime ideal containg $(x_1, \dots, x_d)$, then $\mathfrak{p}$ has height $\geq d$, hence $\mathfrak{p} = \mathfrak{m}$ (for $\mathfrak{p} \subsetneq \mathfrak{m}$, height $\mathfrak{p} <$ height $\underset{d}{\overset{\|}{\mathfrak{m}}}$)

Hence the ideal $(x_1, \dots, x_d)$ is $\mathfrak{m}$-primary.

Suppose $i > 0$ and $x_1, \dots, x_{i-1}$ constructed. Let $\mathfrak{p}_j$ $(1 \leq j \leq s)$ be the minimal prime ideals of $(x_1, \dots, x_{i-1})$ which have height exactly $i-1$.

也即 $\mathrm{Ass}(A/(x_1, \dots, x_{i-1}))$ 中极小元.

Since $i-1 < d = \dim A =$ height $\mathfrak{m} \Rightarrow \mathfrak{m} \neq \mathfrak{p}_j$ $(\forall 1 \leq j \leq s)$

hence $\mathfrak{m} \neq \bigcup_{j=1}^{s} \mathfrak{p}_j$.

Now choose $x_i \in \mathfrak{m} - \bigcup_{j=1}^{s} \mathfrak{p}_j$. Let $\mathfrak{q}$ be any prime containing $(x_1, \dots, x_i)$. then $\mathfrak{q}$ contains some minimal prime ideal $\mathfrak{p}$ of $(x_1, \dots, x_{i-1})$ $\left(\begin{array}{c}(x_1, \dots, x_i) \\ \cap| \\ \mathfrak{p} \subseteq \mathfrak{q}\end{array}\right)$

If $\mathfrak{p} = \mathfrak{p}_j$ for some j, we have $x_i \in \mathfrak{q}$, $x_i \notin \mathfrak{p}$, hence $\mathfrak{p} \subsetneq \mathfrak{q}$ $\Rightarrow$ height $\mathfrak{q} \geq$ height $\mathfrak{p} + 1 = i$.

If $\mathfrak{p} \neq \mathfrak{p}_j$ $(\forall 1 \leq j \leq s)$, then height $\mathfrak{p} >$ height $\mathfrak{p}_j + 1 = i$.

Thus every prime ideal containg $(x_1, \dots, x_i)$ has height $\geq i$. ▦

Example 8.18 $A = k[x_1, \dots, x_n]_{\mathfrak{m}}$, $\mathfrak{m} = (x_1, \dots, x_n)$.

$G_{\mathfrak{m}}(A)$ is a polynomial ring in n-determinants, so its Poincaré series is $\frac{1}{(1-t)^n} \Rightarrow \dim A = n$.

Corollary 8.19 $(A, \mathfrak{m})$ Noether local ring, $k = A/\mathfrak{m}$.

Then $\dim A \leq \dim_k \mathfrak{m}/\mathfrak{m}^2$.

proof If $x_i \in \mathfrak{m}$ $(1 \leq i \leq s)$ are such that their images in $\mathfrak{m}/\mathfrak{m}^2$ f

a basis of $\mathfrak{m}/\mathfrak{m}^2 \Rightarrow \{x_i\}$ generate $\mathfrak{m} \Rightarrow \delta(A) \leq s$

$\Rightarrow \dim A \leq s = \dim_k \mathfrak{m}/\mathfrak{m}^2$.

Corollary 8.20 A: Noether ring, $x_1, \dots, x_r \in A$.

Then every minimal prime ideal $\mathfrak{p}$ belong to $(x_1, \dots, x_r)$ has height ≤

↖ とは$(x_1, \dots, x_r)$の minimal prime i

proof In $A_{\mathfrak{p}}$, $(x_1, \dots, x_r)$ becomes $\mathfrak{p}A_{\mathfrak{p}}$-primary $\mathfrak{p} \in \mathrm{Ass}(A/(x$

$\Rightarrow r \geq \dim A_{\mathfrak{p}} = \text{height } \mathfrak{p}$.

Corollary 8.21 (Krull's principal ideal thm) A: Noether, $x \in A$ which is neither a zero-divisor nor a unit.

~~pf. By 8.20~~

Then every minimal prime ideal $\mathfrak{p}$ of (x) has height 1.

pf. By Corollary 8.20 $\Rightarrow$ height $\mathfrak{p} \leq 1$.

If height $\mathfrak{p} = 0 \Rightarrow \mathfrak{p}$ is a prime ideal belong to 0, hence every element of $\mathfrak{p}$ is a zero divisor

$\Rightarrow x \notin \mathfrak{p}$, which is a contradiction.

Corollary 8.22 A: Noether local ring, $x \in m$ not a zero divisor.

Then $\dim A/(x) = \dim A - 1$.

证. Assume $d = \dim A/(x)$, then $d \leq \dim A - 1$ by 8.14.

反之，若 x_i $(1 \leq i \leq d)$ 是 m 中元素, s.t $(\bar{x}_1, \dots, \bar{x}_d) \subseteq A/x$ is a $m/(x)$-primary ideal. Then $(x, x_1, \dots, x_d)$ is m-primary

$$\Rightarrow \underset{\geq \delta(A) \geq}{d+1 \geq \dim A}$$

▦

Corollary 8.23 $\hat{A}$ = m-adic completion of A. Then $\dim A = \dim \hat{A}$.

证 $A/m^n \cong \hat{A}/\hat{m}^n \Rightarrow \chi_m(n) = \chi_{\hat{m}}(n)$. ▦

Definition 8.24 (A, m) Local Noether local ring. $d = \dim A$

If $(x_1, \dots, x_d)$ generate an m-primary ideal, then we call $x_1, \dots, x_d$ a system of parameters. $\Updownarrow$ $\sqrt{(x_1, \dots, x_d)} = m$.

Prop 8.25 As above, let $q = (x_1, \dots, x_d)$, $\sqrt{q} = m$.

$f(t_1, \dots, t_d)$: homogeneous polynomial of degree s with coeff in A

and assume $f(x_1, \dots, x_d) \in q^{s+1}$ $(s = \deg f)$

Then all coeff of f lie in m.

proof Consider $\alpha: A/q[t_1, \dots, t_d] \twoheadrightarrow G_q(A)$

$t_i \longmapsto \bar{x}_i$ (reduction modulo q)

then $\bar{f}(t_1, \dots, t_d) \in \ker\alpha$

$\Rightarrow A/q[t_1, \dots, t_d]/(\bar{f}) \twoheadrightarrow G_q(A)$.

第一章习题3： If some coeff of f is a unit, then $\bar{f}$ is not

zero divisor.

then $d(G_q(A)) \le d\left(A/q[t_1,\dots,t_d]/(\bar{f})\right)$

$= d\left(A/q[t_1,\dots,t_d]\right) - 1 = d-1$

矛盾!

thus all coeff of f must lie in m.

Corollary 8.26 If $k \subseteq A \longrightarrow A/m \cong k$ and if $x_1,\dots,x_d$ is a system of parameters, then $x_1,\dots,x_d$ are alg-indepent over k.

proof 若存在 $\underset{\underset{A}{\cap}}{k}$ 中多项式 $f \neq 0$ s.t. $f(x_1,\dots,x_d)=0$

write $f = f_s + (\text{higher term})$, where f_s is homo of deg

and $f_s \neq 0$.

Now $f_s(x_1,\dots,x_d)=0$ in q^s/q^{s+1}, $q=(x_1,\dots,x_d)$

By prop 8.25 $\Rightarrow$ f_s has all its coeff in m

But f_s has coeff in k $\Big\}$ $\Rightarrow f_s \equiv 0$ contradi

Thm+Def 8.27 (Regular local ring)

A: Noether local ring of dim d, $m \subseteq A$ maximal ideal and $k=$

We call A a regular local ring if the following equivalent cond

holds: (1) $G_m(A) \cong k[t_1,\dots,t_d]$, where t_i are independent indeter

并不是典范同构

(2) $\dim_k m/m^2 = d = \dim A$

(3) m can be generated by d-elements.

Pf (1)$\Rightarrow$(2) clear. (2)$\Rightarrow$(3) by Nakayama.

(3)$\Rightarrow$(1). Let $\mathfrak{m} = (x_1, \dots, x_d)$. Then $\alpha = k[x_1, \dots, x_d] \longrightarrow G_{\mathfrak{m}}(A)$ is an isom of graded rings by Prop 8.25. ∎

下引理表明：regular local ring 是 integral domain (且 integrally closed).

Lemma 8.28 A: ring. $I \subseteq A$ ideal such that $\bigcap I^n = (0)$.

Suppose that $G_I(A)$ is an integral domain, then A is an integral domain.

proof Let $x, y \neq 0$ in A. Since $\bigcap I^n = 0 \Rightarrow \exists\, r, s \geq 0$ such that

$$x \in I^r - I^{r+1}$$
$$y \in I^s - I^{s+1}$$

Let $\bar{x}, \bar{y}$ be the images of x, y in $G_r(A)$, $G_s(A)$ resp.

Then $\bar{x} \neq 0$, $\bar{y} \neq 0$, hence $\underset{\neq 0}{\bar{x} \cdot \bar{y}} = \overline{x \cdot y} \Rightarrow x \cdot y \neq 0$. ∎

Corollary 8.29 Regular local ring of dim 1 $=$ discrete valuation ring

(以前证明：在 Noether local of dim (条件 下, d.v.r $\Leftrightarrow \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 1$)).

↯ $\exists$ integrally closed domain of dim > 1, which are not regular.

Prop 8.30 A: Noether local ring. Then A is regular iff $\hat{A}$ is regular.

(特别：若 A regular, then $\hat{A}$ is also integrally closed).

proof $\hat{A}$ Noether local, $\dim \hat{A} = \dim A$, $\hat{\mathfrak{m}} = \mathfrak{m}\hat{A}$.

$G_{\mathfrak{m}}(A) = G_{\hat{\mathfrak{m}}}(\hat{A})$. Then apply thm 8.27. ∎

Corollary 8.31 If A Noether local with $k \cong A/\mathfrak{m} \subseteq A$, then

If A regular, then $\hat{A} \cong k[\![X_1, \dots, X_d]\!]$.

apply 8.27.

Example 8.32 $A = k[X_1, \dots, X_n]$, k: field, $\mathfrak{m} = (X_1, \dots, X_n)$.

Then $A_\mathfrak{m}$ is a regular local ring (since $G_\mathfrak{m}(A)$ is a polynomial in n variables).

$\boxed{\text{Atiyah 结束}}$

补充见 Matsumura.

Discussion 8.33 $A \xrightarrow{\phi} B$ homo of rings,

$$\begin{tikzcd} \operatorname{Spec} B \arrow[d, "{}^a\phi"] \\ \mathfrak{p} \in \operatorname{Spec} A \end{tikzcd}$$

$\mathfrak{p} \in \operatorname{Spec} A$, $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$

${}^a\phi^{-1}(\mathfrak{p}) \cong \operatorname{Spec}(B \otimes_A k(\mathfrak{p})) = \operatorname{Spec} B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ fiber $\mathfrak{p}$ of ${}^a\phi$

对 $\mathfrak{q} \in \operatorname{Spec} B$ lying over $\mathfrak{p}$

$$\mathfrak{q} \longmapsto \mathfrak{q}^* = \mathfrak{q}B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$$

$$(B \otimes_A k(\mathfrak{p}))_{\mathfrak{q}^*} = (B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p})_{\mathfrak{q}B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}} \underset{\uparrow}{=} B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q}$$

因为 $(B_\mathfrak{p})_{\mathfrak{q}B_\mathfrak{p}} = B_\mathfrak{q}$.

拓扑上
$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ f^{-1}(y) \arrow[u, phantom, sloped, "\subseteq"] \arrow[r] & y \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$
$\dim X$, $\dim Y$ 与 $\dim f^{-1}(y)$ 关系?

Thm 8.34 $\phi: A \to B$ homo of noetherian rings.

$\mathfrak{q} \in \operatorname{Spec} B$, $\mathfrak{p} = \mathfrak{q} \cap A$. Then

(1) $$\underset{\dim B_\mathfrak{q}}{\underset{\parallel}{\operatorname{ht} \mathfrak{q}}} \le \underset{\dim A_\mathfrak{p}}{\underset{\parallel}{\operatorname{ht} \mathfrak{p}}} + \underset{\dim(B_\mathfrak{q} \otimes k(\mathfrak{p}))}{\underset{\parallel}{\operatorname{ht} B/\mathfrak{p}B}}$$

(2) The equality holds in (1) if the going down thm holds for ϕ (e.g, if ϕ is flat)

(3) if ${}^a\phi : \mathrm{Spec}B \to \mathrm{Spec}A$ is surjective and if the going down thm holds, then we have (i) $\dim B \geq \dim A$

(ii) $ht(I) = ht(IB)$ for any ideal $I \subseteq A$

Thm 8.35 A: Noether ring. Then $\dim A[X_1, \ldots, X_n] = \dim A + n$.

proof enough to prove the case $n=1$.

Put $B = A[X]$. Let $\mathfrak{p} \in \mathrm{Spec}A$ and $\mathfrak{q} \in \mathrm{Spec}B$ such that $\mathfrak{q}$ is maximal among the prime ideals lying over $\mathfrak{p}$.

$$\begin{tikzcd} \mathrm{Spec}B \arrow[d, dash] & \mathfrak{q} \arrow[d, dash] \\ \mathrm{Spec}A & \mathfrak{p} \end{tikzcd}$$

we claim that $ht(\mathfrak{q}/\mathfrak{p}B) = 1$.

In fact, localizing A and B at $A - \mathfrak{p}$, we may assume that $\mathfrak{p}$ is a maximal ideal. Then $B/\mathfrak{p}B = A/\mathfrak{p}[X]$ is a polynomial ring in one variable over a field.

$\Rightarrow B/\mathfrak{p}B$ is a principal ideal domain and every maximal ideal has height $1 \Rightarrow ht(\mathfrak{q}/\mathfrak{p}B) = 1$.

Since B is $\underset{\text{flat}}{\text{free}}$ over A $\Rightarrow$ by thm 8.34(2), $ht(\mathfrak{q}) = ht(\mathfrak{p}) + 1$.

As $\mathrm{Spec}B \twoheadrightarrow \mathrm{Spec}A$ surj $\Rightarrow \dim B = \dim A + 1$. ⊞

Corollary 8.36 k: any field. Then $\dim k[X_1, \dots, X_n] = n$, and ~~the~~ the ideal $(X_1, \dots, X_i)$ is a prime ideal of height i, for $1 \leq i \leq n$.

proof $(0) \subsetneq (X_1) \subsetneq (X_1, X_2) \subsetneq \cdots \subsetneq (X_1, \dots, X_i) \subsetneq \cdots \subsetneq (X_1, \dots, X_n)$ is a prime chain of length n.

Since $\dim k[X_1, \dots, X_n] = n$, the assertion is obvious. ▦

Thm 8.37 A: Noether domain. B/A: f.g domain over A.

$\mathfrak{q} \in \operatorname{Spec} B$, $\mathfrak{p} = \mathfrak{q} \cap A$.

Then we have $ht(\mathfrak{q}) \leq ht\, \mathfrak{p} + tr.deg_A B - tr.deg_{k(\mathfrak{p})} k(\mathfrak{q})$, and the equality holds if A is universally catenary, or if B is a polynomial ring $A[X_1, \dots, X_n]$.

$tr.deg_A B = tr.deg\, \operatorname{Frac} B / \operatorname{Frac} A$, $k(\mathfrak{q})$ = quotient field of $B/\mathfrak{q}$.

2024.05.22 (注意!! 2024.06.12 周三下午期末考试)

2:00-4:00

§9 Cohomology Functors

Freyd-Mitchell embedding thm: If $\mathscr{C}$ is an "abelian category", then $\exists$ ring R and exact fully faithful functor $\mathscr{C} \hookrightarrow \mathrm{Mod}_R$

full sub cat

$\mathrm{Hom}_{\mathscr{C}}(M,N) \cong \mathrm{Hom}_{\mathrm{Mod}_R}(M,N)$.

Definition 9.1 Category of (cochain) complexes.

Let $\mathscr{C}$ be an abelian category. A (cochain) complex in $\mathscr{C}$ is a sequence of morphisms in $\mathscr{C}$:

$$X = X^\bullet : \cdots \to X^{k-1} \xrightarrow{d^{k-1}} X^k \xrightarrow{d^k} X^{k+1} \to \cdots$$

(上同调指标 cohomology index)

such that $d^k \circ d^{k-1} = 0$ for all $k \in \mathbb{Z}$.

call d^k the differential of the complex of X.

A morphism $f: X \to Y$ between complexes is a sequence of morphisms $\{f^k : X^k \to Y^k\}$ such that

$$\begin{tikzcd}
\cdots \arrow[r] & X^{k-1} \arrow[r] \arrow[d, "f^{k-1}"'] \arrow[dr, phantom, "\circlearrowleft"] & X^k \arrow[r] \arrow[d, "f^k"'] \arrow[dr, phantom, "\circlearrowleft"] & X^{k+1} \arrow[r] \arrow[d, "f^{k+1}"] & {} \\
\cdots \arrow[r] & Y^{k-1} \arrow[r] & Y^k \arrow[r] & Y^{k+1} \arrow[r] & {}
\end{tikzcd}$$

$\boxed{\text{These data define the category } C(\mathscr{C}) \text{ of complexes } \cancel{\text{of}} \text{ in } \mathscr{C}}$

Exercise 9.2 $C(\mathscr{C})$ is an abelian category (定义 kernel, image cokernel....)

and $\mathscr{C} \hookrightarrow C(\mathscr{C})$

$A \longmapsto (0 \to A \to 0)$

$\boxed{\text{从而可定义 } C(\mathscr{C}) \text{ 中 exact sequence}}$

Definition 9.3 k-th cohomology functor $H^k(-): C(\mathcal{C}) \longrightarrow \mathcal{C}$.

$\mathcal{C}$: abelian category. $X = (\cdots \to X^{k-1} \xrightarrow{d^{k-1}} X^k \xrightarrow{d^k} X^{k+1} \to \cdots)$ is a cochai
complex in $\mathcal{C}$.

$Z^k(X) := \ker d^k \subseteq X^k$ k-th cocycle object of X

$B^k(X) := \operatorname{Im} d^{k-1} \subseteq X^k$ k-th coboundary object of X

Since $d^k \circ d^{k-1} = 0 \Rightarrow B^k(X) \subseteq Z^k(X) \subseteq X^k$

Define $H^k(X) = \dfrac{Z^k(X)}{B^k(X)} = \dfrac{\ker d^k}{\operatorname{Im} d^{k-1}}$, k-th cohomology of the complex

Any ~~functor~~ morphism $X \xrightarrow{f} Y$ in $C(\mathcal{C})$ induces a family of functors
(hence $df = fd$)

$$H^k(f): H^k(X) \longrightarrow H^k(Y)$$

Reason:

$$\begin{tikzcd}
X^{k-1} \arrow[r, "d^{k-1}"] \arrow[d, "f^{k-1}"] \arrow[dr, phantom, "\circlearrowleft"] & X^k \arrow[r, "d^k"] \arrow[d, "f^k"] \arrow[dr, phantom, "\circlearrowleft"] & X^{k+1} \arrow[d, "f^{k+1}"] \\
Y^{k-1} \arrow[r, "d^{k-1}"] & Y^k \arrow[r, "d^k"'] & Y^{k+1}
\end{tikzcd}$$

$d^k_Y(f^k(\ker d^k_X)) = f^{k+1} d^k(\ker d^k$
$= 0$
$\Rightarrow f^k(\ker d^k_X) \subseteq \ker d^k_Y$.

X exact at $X^k \Leftrightarrow H^k(X) = 0$

$X = X^\bullet$ is an exact sequence $\Leftrightarrow H^k(X) = 0$ for all k.

Example 9.4 (Split "complex" into short exact sequence)

$A \xrightarrow{d} B$ a morphism in $\mathcal{C}$ $\Rightarrow$ two short exact seq associated to the morphism d $\begin{cases} 0 \to \ker d \to A \to \operatorname{Im} d \to 0 \\ 0 \to \operatorname{Im} d \to B \to \operatorname{coker} d \to 0 \end{cases}$

If $X = X^\bullet$ is a complex in $\mathcal{C}$, then we can split it into a family of exact sequences:

$$0 \to B^k(X) \to Z^k(X) \to H^k(X) \to 0$$

$$0 \to Z^{k-1}(X) \to X^{k-1} \to B^k(X) \to 0$$

$$0 \to B^k(X) \to X^k \to \operatorname{coker} d^{k-1} \to 0$$

$$\begin{tikzcd}
0 \arrow[r] & H^k(X) \arrow[r] \arrow[d, equal] & \operatorname{coker} d^{k-1} \arrow[r, "d^k"] \arrow[d, equal] & Z^{k+1}(X) \arrow[r] & H^{k+1}(X) \arrow[d] \\
 & \dfrac{\ker d^k}{\operatorname{Im} d^{k-1}} \arrow[r] & \dfrac{X^k}{\operatorname{Im} d^{k-1}} \arrow[r] & \ker d^{k+1} \arrow[r] & \dfrac{\ker d^{k+1}}{\operatorname{Im} d^k} \arrow[r] & 0 \\
 & & & & 0 &
\end{tikzcd}$$

Remark 9.5 (Homology index)

A chain complex in an abelian category $\mathscr{C}$ is a sequence

$$X = X_\bullet : \cdots \to X_{k+1} \xrightarrow{d_{k+1}} X_k \xrightarrow{d_k} X_{k-1} \to \cdots$$

such that $d_k \circ d_{k+1} = 0 \ (\forall k)$,

Similar to define $Ch(\mathscr{C})$ the category of chain complexes in $\mathscr{C}$.

If we put $X^k = X_{-k}$, $d^k = d_{-k}$, then $X^\bullet$ is a cochain complex.

The object $H_k(X_\bullet) = \dfrac{\operatorname{Ker} d_k}{\operatorname{Im} d_{k+1}}$ is called the k-th homology of $X_\bullet$

Example 9.6 Singular chain complex

$$\begin{tikzcd}
\mathbb{Z}\mathrm{Sing}_\bullet : \mathrm{Top}^{CW} \arrow[r] & Ch(\mathcal{A}b) \arrow[l, bend right, dashed]
\end{tikzcd}$$

For $X \in \mathrm{Top}^{CW}$, $\mathbb{Z}\mathrm{Sing}_n(X) = \mathbb{Z}[\mathrm{Hom}_{\mathrm{conti}}(|\Delta^n|, X)]$

$$\mathrm{Sing}_\bullet(X) : \cdots \longrightarrow \mathbb{Z}\mathrm{Sing}_n(X) \xrightarrow{\sum_{i=0}^{n}(-1)^i d_i} \mathbb{Z}\mathrm{Sing}_{n-1}(X) \to \cdots$$

㊟ Top 并不是 Abel 范畴. 可将其嵌入 abelian cat:

$$\mathrm{Top}^{CW} \hookrightarrow \text{"Condensed Set"}.$$

问题：对两个 <u>complex</u> X与Y，它们何时具有相同的 cohomology？

如果 X，Y 来自拓扑空间，当相应的拓扑空间同伦时，X与Y 具有相同的 singular homology：

↪ 将定义转移到 Ch(Ab)中，得到 complex 同伦的定义.

<u>Definition 9.7</u> $\mathcal{C}$: abelian category.

$X \xrightarrow{f} Y$ and $X \xrightarrow{g} Y$ two morphisms in $C(\mathcal{C})$.

We say f is homotopic to g ($\overset{\text{记为}}{f \sim g}$) if there exist morphisms $s^n: X^n \longrightarrow Y^{n-1}$ in $\mathcal{C}$ such that $f^n - g^n = s^{n+1} \circ d_X^n + d_Y^{n-1} \circ s^n$

图示：

$$\begin{tikzcd}
\cdots \arrow[r] & X^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, shift right, "f^{n-1}"'] \arrow[d, shift left, "g^{n-1}"] & X^n \arrow[r, "d^n"] \arrow[d, shift right, "f^n"'] \arrow[d, shift left, "g^n"] \arrow[dl, "s^n"'] & X^{n+1} \arrow[r] \arrow[d, shift right, "f^{n+1}"'] \arrow[d, shift left, "g^{n+1}"] \arrow[dl, "s^{n+1}"'] & \cdots \\
\cdots \arrow[r] & Y^{n-1} \arrow[r, "d^{n-1}"'] & Y^n \arrow[r, "d^n"'] & Y^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

We say that $f: X \longrightarrow Y$ is an homotopy equivalene if there is a map $h: Y \longrightarrow X$ such that $h \circ f$ and $f \circ h$ are homotopic to id_X and id_Y respective

If there is a homotopy equivalene between X and Y, then we say X is homotopic to Y (written $X \sim Y$).

<u>Lemma 9.8</u> If f and g are homotopic, then ~~it~~ they induce the same maps $H^n(f) = H^n(g): H^n(X) \longrightarrow H^n(Y) \quad \forall n \in \mathbb{Z}$

In particular, if $X \sim Y$, then $H^n(X) \cong H^n(Y)$ for all $n \in \mathbb{Z}$.

Proof If $f \sim g \Rightarrow f-g \sim 0$.

Thus we may assume $g=0$ $(f \sim 0)$.

Then for each n, we have $f^n = s^{n+1} \circ d_X^n + d_Y^{n-1} \circ s^n$.

$$H^n(f) : \frac{\operatorname{Ker} d_X^n}{\operatorname{Im} d_X^{n-1}} \longrightarrow \frac{\operatorname{Ker} d_Y^n}{\operatorname{Im} d_Y^{n-1}}.$$

Since
$$\begin{tikzcd}
\operatorname{Ker} d_X^n \arrow[r] & X^n \arrow[rr, "s^{n+1}\circ d_X^n"] \arrow[dr] & & Y^n \\
& & X^{n+1} \arrow[ur, "s^{n+1}"] &
\end{tikzcd}$$
is zero and

$$\begin{tikzcd}
X^n \arrow[rr, "d_Y^{n-1}\circ s^n"] \arrow[d] & & Y^n \\
\operatorname{Im} d_Y^{n-1} \arrow[urr] & &
\end{tikzcd}$$

$$\Rightarrow \operatorname{Im}(\operatorname{Ker} d_X^n \to X^n \xrightarrow{\overset{s^{n+1}\circ d_X^n + d_Y^{n-1}\circ s^n}{\overset{\parallel}{f^n}}} Y^n)$$
$$\subseteq \operatorname{Im} d_Y^{n-1}$$
$$\Rightarrow H^n(f) = 0.$$ ∎

Exactness 9.9 Let $F : \mathscr{C} \longrightarrow \mathscr{D}$ be a functor between abelian categories. Then it induces a functor

$$F : C(\mathscr{C}) \longrightarrow C(\mathscr{D})$$

Does F preserve cohomology? when?

This is related to the exactness of F.

Recall that: F is exact $\Leftrightarrow$ F is both left exact and right exact.

$\overset{\text{习题}}{\Leftrightarrow}$ F preserves finite limits and finite colimits.

In particular, exact functors preserve kernel/cokernel/image/coimage

$\Rightarrow$ exact functors preserve cohomologies, i.e.,

$$H^n(F(X)) \cong F(H^n(X)) \qquad \forall\, X \in C(\mathscr{C}),\ \forall n.$$

Example 9.10 $\mathcal{C}$: abelian category and $X \in \mathrm{Ob}\,\mathcal{C}$.

(1) $\mathrm{Hom}_{\mathcal{C}}(X,-): \mathcal{C} \longrightarrow Ab$ left exact funtor

$\mathrm{Hom}_{\mathcal{C}}(X,-)$ exact $\overset{\text{def}}{\Longleftrightarrow}$ X projective

(2) $\mathrm{Hom}_{\mathcal{C}}(-,X): \mathcal{C}^{op} \longrightarrow Ab$ left exact

$\mathrm{Hom}_{\mathcal{C}}(*,X)$ exact $\overset{\text{def}}{\Longleftrightarrow}$ X injective.

(3) exact test.

$M' \longrightarrow M \longrightarrow M'' \to 0$ exact $\Leftrightarrow$ $\forall X \in \mathcal{C}$,
$0 \to \mathrm{Hom}(M'',X) \longrightarrow \mathrm{Hom}(M,X) \to \mathrm{Hom}(M',$
exact

$0 \to N' \to N \to N''$ exact $\Leftrightarrow$ $\forall X \in \mathcal{C}$,
$0 \to \mathrm{Hom}(X,N') \to \mathrm{Hom}(X,N) \to \mathrm{Hom}(X,N'')$
exact

(4) $\mathcal{C} = \mathrm{Mod}_R$, R: ring.

$X \in \mathrm{Mod}_R$.

$-\otimes_R X : \mathrm{Mod}_R \longrightarrow Ab$ (left adjoint to $\mathrm{Hom}_{Ab}(X,-)$) is rig
e

$-\otimes_R X$ exact $\overset{\text{def}}{\Longleftrightarrow}$ X flat.

Now, we want to prove that $\{H^n : C(\mathcal{C}) \longrightarrow \mathcal{C}\}_n$ is a cohomology δ-funtor, i.e., each short exact sequence in $C(\mathcal{C})$ ind
a long exact sequence in $\mathcal{C}$ (which is also funtorial).

Proposition 9.11 Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $C(\mathscr{C})$.

Then there is a canonical long exact sequence in $\mathscr{C}$

$$\begin{tikzcd}
\cdots \arrow[r] & H^n(X) \arrow[r] & H^n(Y) \arrow[r] & H^n(Z) \arrow[dll, "\delta"] \\
& H^{n+1}(X) \arrow[r] & H^{n+1}(Y) \arrow[r] & H^{n+1}(Z) \arrow[r] & \cdots
\end{tikzcd}$$

Moreover, if

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram with exact rows in $C(\mathscr{C})$,

then the diagram

$$\begin{tikzcd}
H^n(Z) \arrow[r, "\delta"] \arrow[d] & H^{n+1}(X) \arrow[d] \\
H^n(Z') \arrow[r, "\delta"] & H^{n+1}(X')
\end{tikzcd}$$

commutes.

proof "以后将 differential map 统一记为 d".

By snake Lemma, we have a commutative diagram with exact rows:

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Z^n(X) \arrow[r] \arrow[d] & Z^n(Y) \arrow[r] \arrow[d] & Z^n(Z) \arrow[d] \arrow[dddll, dashed, bend left] & \\
0 \arrow[r] & X^n \arrow[r] \arrow[d] & Y^n \arrow[r] \arrow[d] & Z^n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X^{n+1} \arrow[r] \arrow[d] & Y^{n+1} \arrow[r] \arrow[d] & Z^{n+1} \arrow[r] \arrow[d] & 0 \\
& \frac{X^{n+1}}{dX^n} \arrow[r] \arrow[d] & \frac{Y^{n+1}}{dY^n} \arrow[r] \arrow[d] & \frac{Z^{n+1}}{dZ^n} \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

We have a commutative diagram with exact rows:

$$\begin{tikzcd}
 & \frac{X^n}{dX^{n-1}} \arrow[r] \arrow[d, "d"] & \frac{Y^n}{dY^{n-1}} \arrow[r] \arrow[d] & \frac{Z^n}{dZ^{n-1}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z^{n+1}(X) \arrow[r] & Z^{n+1}(Y) \arrow[r] & Z^{n+1}(Z) &
\end{tikzcd}$$

By Snake lemma again, we get an exact sequence

$$H^n(X) \to H^n(Y) \to H^n(Z) \xrightarrow{\delta} H^{n+1}(X) \to H^{n+1}(Y) \to H^{n+1}(Z)$$

"δ的函子性"(习题)

▨

<u>Remark 9.12</u> A more fancy way to prove prop 9.11 is to use mapping con

and shift functors (参见作业题).

<u>Remark 9.13</u> Similar result in algebraic topology.

$E \xrightarrow{\pi} B$ Serre fibration of top spaces.

F = homotopy fiber of π.

Then there is a long exact sequence of fundamental groups

$$\cdots \to \pi_{n+1}(B) \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots$$

<u>Recall 9.14 (cohomological (δ-)functor)</u>

Let $\mathscr{C}$ and $\mathscr{D}$ be abelian categories. A covariant cohomological functor from $\mathscr{C}$ to $\mathscr{D}$ is a family of covariant functors $T=(T^i)_{i\in\mathbb{N},\, \text{or } i\in\mathbb{Z}}$ together with morphisms $\delta^i: T^i(C) \to T^{i+1}(A$

for each short exact sequence $0 \to A \to B \to C \to 0$, satisfying the following conditions:

(1) Associated to the above short exact sequence, we have a long exact sequence

$$T^i(A) \longrightarrow T^i(B) \to T^i(C) \xrightarrow{\delta^i} T^{i+1}(A) \to T^{i+1}(B) \to T^{i+1}(C)$$

(对 $T=(T^i)_{i\in\mathbb{N}}$, 可设 $T^{-1}=0$).

(2) For any morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

the following diagram commutes for each i:

$$\begin{tikzcd}
T^i(C) \arrow[r, "\delta^i"] \arrow[d] & T^{i+1}(A) \arrow[d] \\
T^i(C') \arrow[r, "\delta^i"] & T^{i+1}(A')
\end{tikzcd}$$

接下来目标: For each left (resp. right) exact functor $F: \mathcal{C} \to \mathcal{D}$ between abelian categories (~~[illegible]~~, any additive functor or non-additive functor), we will define its right derived functors $\{R^nF: \mathcal{C} \to \mathcal{D}\}_{n\in\mathbb{N}}$ (resp. left derived functors $\{L_nF: \mathcal{C} \to \mathcal{D}\}_{n\in\mathbb{N}}$) such that $\{R^nF\}$ (resp. $\{L_nF\}$) forms a cohomological (resp. homological) (δ-) functor.

方式: 利用 injective/projective resolutions.

Recall 9.15 (Properties of injective/projective objects) 上半学期已证

$\mathcal{C}$: abelian category.

(1) $I \in \mathcal{C}$ injective $\Leftrightarrow \mathrm{Hom}_{\mathcal{C}}(-, I): \mathcal{C}^{op} \to Ab$ exact functor

$\Leftrightarrow \forall\ 0 \to X \xrightarrow[mono]{} Y$, $\mathrm{Hom}_{\mathcal{C}}(Y, I) \xrightarrow[surj]{} \mathrm{Hom}_{\mathcal{C}}(X, I) \to$

$\Leftrightarrow$ any diagram

$$\begin{tikzcd} X \arrow[r, hook, "mono"] \arrow[d] & Y \arrow[dl, dashed] \\ I & \end{tikzcd}$$

has a solution for the dotted arrow.

(2) $P \in \mathcal{C}$ projective $\Leftrightarrow \mathrm{Hom}_{\mathcal{C}}(P, -)$ exact functor

$\Leftrightarrow \forall\ X \xrightarrow[epi]{} Y \to 0$, $\mathrm{Hom}_{\mathcal{C}}(P, X) \xrightarrow[surj]{} \mathrm{Hom}_{\mathcal{C}}(P, Y) \to$

$\Leftrightarrow$ any diagram

$$\begin{tikzcd} X \arrow[r, two heads] & Y \arrow[r] & 0 \\ & P \arrow[u] \arrow[ul, dashed] & \end{tikzcd}$$

has a solution for the dotted arrow

(3) If $0 \to X \xrightarrow{f} Y \to Z \to 0$ exact with X injective, then the sequ splits ($Y \cong X \oplus Z$), in particular, Y is injective iff Z is injective

Similar for projective.

(4) Baer's Criterion.

$M \in Mod_R$. M injective $\Leftrightarrow$ $\forall$ ideal $I \subseteq R$, any morphism $I \to$ can be exteded to a morphism $R \to M$

(特别取 $I=(x)$,

$$\begin{tikzcd} I \arrow[r] \arrow[d] & M \\ R \arrow[ur, dashed] & \end{tikzcd}$$

"$\frac{m}{x}$"

If R is a principal ideal domain (e.g $R = \mathbb{Z}$), then $\left(\begin{array}{l} M \text{ is injective} \\ \Updownarrow \\ M \text{ is divisible} \end{array} \right.$

i.e., $\forall r \neq 0$ in R, $\forall m \in M$, $\exists n$ s.t. $m = rn$ "$\frac{m}{r}$" 有意义.

(5) $M \in \mathrm{Mod}_R$ proj $\Longleftrightarrow$ M is a direct summand of a free R-mod.

(6) $M \in \mathrm{Mod}_R$ is R-flat iff its Pontrjagin dual $M^* = \mathrm{Hom}_{Ab}(M, \mathbb{Q}/\mathbb{Z})$ is an injective R-module.

(7) $\mathbb{Q}/\mathbb{Z} \in Ab$ is injective.

For any com. ring R, $\mathrm{Hom}_{Ab}(R, \mathbb{Q}/\mathbb{Z})$ is an injective R-module

$$(\mathrm{Hom}_R(-, \mathrm{Hom}_{Ab}(R, \mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{Ab}(-, \mathbb{Q}/\mathbb{Z})).$$

(8) If an abelian category $\mathcal{C}$ has enough injectives, then any object in $\mathcal{C}$ has an <u>injective resolution</u>. Same for projective objects.

e.g. $\mathrm{Mod}_{\mathbb{Z}} = Ab$, Mod_R has enough injectives.

2024.05.29

Definition 9.16 $F: \mathcal{C} \to \mathcal{D}$ left exact covariant functor between abelian categories. Assume that $\mathcal{C}$ has enough injective objects. We define the right derived functors $R^iF: \mathcal{C} \longrightarrow \mathcal{D}$ ($i = 0, 1, \dots$) of F as follows:

$\forall A \in \mathcal{C}$, choose injective resolution $I_A^{\cdot} \xleftarrow{q.is} A$, then

$$R^iFA := H^i(F(I_A^{\cdot})) \quad \forall i \geq 0.$$

In the following proposition 9.17 and prop 9.18 imply that the definition of R^iFA is independent of the choice of $I_A^{\cdot}$ and that R^iFA is natural in A, i.e., $R^iF : \mathscr{C} \to \mathscr{D}$ is a well-defined functor.

<u>prop 9.17</u> Let $\mathscr{C}$ be an abelian category. Suppose

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & A^0 \arrow[r] & A^1 \arrow[r] & \cdots & \text{exact} \\
0 \arrow[r] & B \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots & (\text{complex}, I^i \text{ injective})
\end{tikzcd}$$

with rows are complexes, with first sequence exact, and each I^i injective.
Then $\exists$ a morphism of complexes $f^\cdot : A^\cdot \to I^\cdot$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & A^0 \arrow[r] \arrow[d, "f^0"] & A^1 \arrow[r] \arrow[d, "f^1"] & \cdots \\
0 \arrow[r] & B \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots
\end{tikzcd}$$

commutes. We call such $f^\bullet$ a morphism extending f.

If $g^\cdot : A^\cdot \to I^\cdot$ is another morphism extending f, then $f^\bullet$ and $g^\bullet$ are hom

<u>proof</u> Since I^0 is injective and A, is a subobject of A^0, we have

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & A^0 \arrow[d, "\exists f^0"] \\
0 \arrow[r] & B \arrow[r] & I^0
\end{tikzcd}$$

Suppose that we have defined f^i for any $i<n$ such that

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & A^0 \arrow[r] \arrow[d, "f^0"'] \arrow[dr, phantom, "\circlearrowleft"] & A^1 \arrow[r] \arrow[d, "f^1"'] & \cdots \arrow[r] & A^{n-2} \arrow[r] \arrow[d, "f^{n-2}"'] \arrow[dr, phantom, "\circlearrowleft"] & A^{n-1} \arrow[r] \arrow[d, "f^{n-1}"] & A^n \\
0 \arrow[r] & B \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots \arrow[r] & I^{n-2} \arrow[r] & I^{n-1} \arrow[r] & I^n
\end{tikzcd}$$

<u>下面构造 $f^n : A^n \to I^n$.</u>

首先 $A^{n-1} \xrightarrow{f^{n-1}} I^{n-1} \longrightarrow I^n$ vanishes on $\mathrm{Im}(A^{n-2} \to A^{n-1})$, it induces a morphism

$$\begin{tikzcd} \operatorname{coker}(A^{n-2} \arrow[r] & A^{n-1}) \arrow[r] \arrow[d, hook] & I^n \\ & A^n & \end{tikzcd}$$

$\Rightarrow \exists f^n : A^n \to I^n$ such that

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & A^0 \arrow[r] \arrow[d, "f^0"] & {} & \arrow[r] & A^n \arrow[d, "f^n"] \\ 0 \arrow[r] & B \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots \arrow[r] & I^n \end{tikzcd}$$ Commutes.

In this way, we get a morphism $f^\cdot : A^\cdot \to I^\cdot$ extending $f : A \to B$.

留作习题

Suppose $g^\cdot : A^\cdot \to I^\cdot$ is another morphism extending $f : A \to B$

Let $I^{-1} = 0$ and define $h^0 : A^0 \to I^{-1}$ to be the zero morphism. Suppose we have defined $h^i : A^i \to I^{i-1}$ for any $i \leq n$ such that $f^i - g^i = h^{i+1} d^i + d^{i-1} h^i$

we have $(f^n - g^n - d^{n-1} h^n) d^{n-1} = d^{n-1}(f^{n-1} - g^{n-1}) - d^{n-1} h^n d^{n-1}$

$$= d^{n-1}(h^n d^{n-1} + d^{n-2} h^{n-1}) - d^{n-1} h^n d^{n-1} = 0.$$

$\Rightarrow f^n - g^n - d^{n-1} h^n$ vanishes on $\operatorname{Im}(A^{n-1} \to A^n)$.

thus $\exists$

$$\begin{tikzcd} \operatorname{Coker}(A^{n-1} \arrow[r] & A^n) \arrow[r, "\text{induces}"] \arrow[d, hook] & I^n \\ & A^{n+1} \arrow[ur, dashed, "h^{n+1}"'] & \end{tikzcd}$$

then $h^{n+1} d^n + d^{n-1} h^n = f^n - g^n$. ▦

<u>Proposition 9.18</u> Let $\mathcal{C}$ be an abelian category. Then any two injective resolutions of an object $A \in \mathcal{C}$ are homotopy equivalent.

<u>proof</u> Let $I^\cdot$ and $J^\cdot$ be two injective resolutions of A.

By Prop 9.17 $\Rightarrow \exists f : I^\cdot \to J^\cdot$ extending id_A, $\exists g : J^\cdot \to I^\cdot$ extending id_A.

Note that both $g^\bullet f^\bullet : I^\bullet \longrightarrow I^\bullet$ and $\mathrm{id}_{I^\bullet}$ are morphisms extending id_A.

By prop 9.17 $\Rightarrow g^\bullet f^\bullet \underset{\text{homotopy}}{\sim} \mathrm{id}_{I^\bullet}$

Similarly, $\exists$ homotopy $f^\bullet g^\bullet \sim \mathrm{id}_{J^\bullet}$

$\Rightarrow I^\bullet$ and $J^\bullet$ are homotopy equivalent. ▦

$\boxed{\text{Back to } \cancel{\text{def}}\ 9.16}$

By prop 9.18, $R^iFA := H^i(F(I_A^\bullet))$ is independent of the choice of $I_A^\bullet$

(要求 F additive).

For $A \xrightarrow{f} B$, can choose

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & I_A^\bullet \arrow[d] \\
B \arrow[r] & I_B^\bullet
\end{tikzcd}$$

then $\exists$ $R^iFA \longrightarrow R^iFB$ …… ▦

以下性质将用于 "derive a long exact seq for $\{R^iF\}$"

也将用于定义 hyper-derived funtors

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "R^iF"] \arrow[d] & \mathcal{D} \\
\mathcal{C}^+(\mathcal{C}) \arrow[ur, "\mathbb{R}^iF"'] &
\end{tikzcd}$$

<u>prop 9.19 (Horseshoe Lemma)</u>

$0 \to A \to B \to C \to 0$ exact seq in an abelian category $\mathcal{C}$.

$A \to I^\bullet$, $C \to J^\bullet$ injective resolutions.

Then there exists an exact sequence $0 \to B \to I^0 \oplus J^0 \longrightarrow I^1 \oplus J^1 \to \cdots$

such that

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^0 \arrow[r] \arrow[d] & I^0\oplus J^0 \arrow[r] \arrow[d] & J^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^1 \arrow[r] \arrow[d] & I^1\oplus J^1 \arrow[r] \arrow[d] & J^1 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

where the morphisms $I^\bullet \longrightarrow I^\bullet \oplus J^\bullet$ and $I^\bullet \oplus J^\bullet \longrightarrow J^\bullet$ are the canonical ones. ▦

(证明见PDF)

Discussion 9.20 $F: \mathscr{C} \longrightarrow \mathscr{D}$ left exact functor between abelian categories. Assume that $\mathscr{C}$ has enough injective objects.

Fact 1 $R^0 F = F$

Since F is left exact, the sequence $0 \to FA \to FI_A^0 \to FI_A^1$ exact

$\Rightarrow FA = H^0(F(I_A^\bullet)) = R^0 FA$.

Fact 2 For any injective object $A \in \mathscr{C}$, $R^i FA = 0$ for any $i \geq 1$.

In fact, $0 \to \underline{A \to 0 \to 0}$ is an injective resolution of A,

[redacted]

thus $\Rightarrow \cdots$.

Fact 3 Long exact sequence associated to a short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in $\mathscr{C}$.

By Horseshoe Lemma, $\exists$ short exact seq of complexes

$$0 \to I_A^{\bullet} \to I_B^{\bullet} \to I_C^{\bullet} \to 0$$

such that $I_A^{\bullet}$, $I_B^{\bullet}$ and $I_C^{\bullet}$ are injective resolutions of A, B and C resp

and for each i, we have $I_B^i \simeq I_A^i \oplus I_C^i$.

Since F is additive, $\Rightarrow 0 \to F(I_A^{\bullet}) \to F(I_B^{\bullet}) \to F(I_C^{\bullet}) \to 0$ is a split short exact seq of complexes.

Then apply H^i get morphisms $\delta^i : R^iF(C) \to R^{i+1}F(A)$. together

a long exact sequence (by 9.11)

$$\cdots \to R^iFA \to R^iFB \to R^iFC \xrightarrow{\delta^i} R^{i+1}FA \to$$

(which is functorial w.r.t $0 \to A \to B \to C \to 0$ 于[illegible])

$\rightsquigarrow \{R^iF\}_{i \geq 0}$ cohomological δ-functor.

<u>Definition 9.21</u> (acyclic objects)

We call $J \in \mathcal{C}$ is F-acyclic iff $R^iF(J) = 0 \ \forall i \geq 1$.

(Thus injective objects are F-acyclic)

An F-acyclic resolution of $A \in \mathcal{C}$ is a complex of the form $0 \to J^0 \to J^1 \to \cdots$ together with a monomorphism $A \to J^0$ such that each J^i is F-acyclic, and the sequence $0 \to A \to J^0 \to J^1 \to \cdots$ is exact.

可用"F-acyclic resolution"去计算 R^iFA.

Proposition 9.22 $\mathcal{C} \xrightarrow{F} \mathcal{D}$ left exact / ab. categories.

$\mathcal{C}$ has enough injective objects.

$J^\bullet$ = F-acyclic resolution of A. Then $R^iF(A) \cong H^i(F(J^\bullet))$.

proof $$\left.\begin{array}{l} 0 \to A \to J^0 \to Z^1(J^\bullet) \to 0 \text{ exact} \\ R^iF(J^0) = 0 \ \forall i \geq 1 \end{array}\right\} \Rightarrow \begin{array}{c} R^{i-1}F(Z^1(J^\bullet)) \\ \wr\text{s} \\ R^iFA \quad \forall i \geq 2. \end{array}$$

$0 \to Z^1(J^\bullet) \to J^1 \to Z^2(J^\bullet) \to 0$ exact

$R^iF(J^1) = 0 \ \forall i \geq 1$

$\Rightarrow R^{i-2}F(Z^2(J^\bullet)) \cong R^{i-1}F(Z^1(J^\bullet)) \cong R^iF(A), \forall i \geq 3.$

$\Rightarrow R^iF(A) = R^{i-1}F(Z^1(J^\bullet)) = \cdots = R^1F(Z^{i-1}(J^\bullet)) \ \forall i \geq 1$

From $0 \to Z^{i-1}(J^\bullet) \to J^{i-1} \to Z^i(J^\bullet) \to 0$ exact together

with $R^1F(J^{i-1}) = 0$

$\Rightarrow 0 \to F(Z^{i-1}(J^\bullet)) \to F(J^{i-1}) \to F(Z^i(J)) \to R^1F(Z^{i-1}(J^\bullet)) \to 0$

$$\Rightarrow R^iF(A) = R^1F(Z^{i-1}(J)) = \frac{F(Z^i(J))}{\mathrm{Im}(F(\overset{J^{i-1}}{\cancel{Z^{i-1}(J)}}) \to F(\overset{Z^i(J)}{\cancel{J^{i-1}}}))} \quad (*)$$

Since $0 \to Z^i(J) \to J^i \to J^{i+1}$ exact and F left exact

$\Rightarrow 0 \to F(Z^i(J)) \to F(J^i) \to F(J^{i+1})$ exact

$\Rightarrow F(Z^i(J)) = \ker(F(J^i) \to F(J^{i+1}))$

$\cdot \ \mathrm{Im}(F(J^{i-1}) \to F(Z^i(J))) = \mathrm{Im}(F(J^{i-1}) \to F(J^i))$

(*)
$\Rightarrow R^iFA = H^i(F(J^\cdot)) \quad \forall i \geq 1.$

(*) ~~$F(Z^i(J))$~~

~~$\mathrm{Im}(F(Z^{i-1}(J)) \to F(J^{i+1}))$~~

Since F is left exact, we have $R^0FA \cong H^0F(J^\cdot)$ ▦

Remark 9.23 (Dimension shifting)

$0 \to A \to B \to C \to 0$ exact with B F-acyclic, then we have

$$R^iFC \cong R^{i+1}F(A) \quad \forall i \geq 1.$$

More generally, if $0 \to A \to B_0 \to B_1 \to \cdots \to B_m \to C \to 0$ is exact. such that B_i are F-acyclic objects, then we have

$$R^iFC \cong R^{i+m+1}F(A) \quad \forall i \geq 1.$$

问题9.24 $F: \mathscr{C} \to \mathscr{D}$ left exact with $\mathscr{C}$ enough injective objec

如果从F出发，利用制作的方式定义了一组 cohomological δ-functor $T=(T$

且 $T^0 = F$. 在哪些条件下，可断言 $(T^i)_{i\geq 0}$ 为 F 的右导出函子，也

即说 $T^i \cong R^iT^0 = R^iF \quad \forall i \geq 0.$

Definition 9.25 (Universal cohomological δ-functor)

We say a cohomological functor $T=(T^i)$ is universal if for any cohomological functor $T'=(T'^i)$ and any natural transformation $f^0: T^0 \longrightarrow T'^0$, there exists a unique family of natural transformation $f^i: T^i \longrightarrow T'^i$ $(i \geq 1)$ such that for any short exact sequence $0 \to A \to B \to C \to 0$, the following diagram commutes for each i

$$\begin{tikzcd}
T^i(C) \arrow[r, "\delta^i"] \arrow[d, "f^i_C"'] & T^{i+1}(A) \arrow[d, "f^{i+1}_A"] \\
T'^i(C) \arrow[r, "\delta^i"] & T'^{i+1}(A)
\end{tikzcd}$$

(习题: $\{H^n: \mathcal{C}(\mathcal{A}) \to \mathcal{A}\}$ universal δ-functor)

Theorem 9.26 $F: \mathcal{C} \longrightarrow \mathcal{D}$ left exact between ab. Cat with $\mathcal{C}$ enough injective objects. Then

(1) $(R^iF)_{i\geq 0}$ is a universal cohomological functor

(2) If $T=(T^i)_{i\geq 0}$ is a covariant $\boxed{\text{universal}}$ cohomological functor, then T^0 is left exact and $T^i \cong R^i(T^0)$ $\forall i \geq 0$

T^0的右导出.

特别, 若 $T^0=F$, then $T^i \cong R^iF$.

((2) 是 (1) 的推论, 下证明 (1)).

2024.06.03

首先, R^iF is effaceable in the following sense:

Definition 9.27 (effaceable and coeffaceable)

A functor $F:\mathscr{C}\longrightarrow\mathscr{D}$ is called effaceable if for any object $A\in\mathscr{C}$, $\exists$ monomorphism $A\overset{u}{\hookrightarrow}M$ such that $F(u)=0$.

(coeffaceable if $\forall A\in\mathscr{C}$, $\exists$ epimorphism $M\xrightarrow{u}A$ s.t $F(u)=0$)

注| 对于 R^iF, 当 F left exact 时, 取 M injective, 可知 R^iF effaceable.

注| 对于 L_iF, 当 F right exact 时, 取 M projective, 可知 L_iF coeffaceable.

以下结论表明 $\{R^iF\}$ is a universal cohomological functor ($\Rightarrow$ thm 9.26)

Proposition 9.28 Let $T=(T^i):\mathscr{C}\longrightarrow\mathscr{D}$ be a covariant cohomolo functor between abelian categories. If T^i is effaceable for any $i\geq$ then T is universal.

proof Let $T'=(T'^i)$ be a cohomological functor and $f^0:T^0\longrightarrow T'^0$ a natural transformation.

Suppose that we have shown that: there is a unique family of natural transf $f^i:T^i\longrightarrow T'^i$ $(0\leq i\leq n-1)$ such that for any short exact sequence $0\to A\to B\to C\to 0$, the following diagram commutes whenever $0\leq i\leq$

$$\begin{tikzcd}
T^iC \arrow[r, "\delta^i"] \arrow[d, "f_C^i"'] & T^{i+1}A \arrow[d, "f_A^{i+1}"] \\
T'^iC \arrow[r, "\delta^i"] & T'^{i+1}A
\end{tikzcd}$$

下从构造 $f^n: T^n \longrightarrow T'^n$ 且与 δ^i 交换:

$\forall A \in \mathscr{C}$, choose mono $u: A \hookrightarrow M$ such that $T^n(u) = 0$.

$0 \to A \to M \to M/A \to 0$ exact.

$$\begin{tikzcd}
T^{n-1}M \arrow[r] \arrow[d, "f_M^{n-1}"] & T^{n-1}(M/A) \arrow[r] \arrow[d, "f_{M/A}^{n-1}"'] & T^nA \arrow[r, "0", "\text{zero mo}"'] \arrow[d, dashed, "\exists f_A^n = f_{A,u}^n"] & 0 & \text{exact} \\
T'^{n-1}M \arrow[r] & T'^{n-1}(M/A) \arrow[r] & T'^n(A) & \text{exact}
\end{tikzcd}$$

$\Rightarrow$

<u>Claim</u> $f_A^n = f_{A,u}^n$ is independent of the choice of $u: A \hookrightarrow M$ ~~$u: A \hookrightarrow M$~~ with $T^n(u) = 0$.

Let $v: A \hookrightarrow N$ be another mono with $T^n v = 0$.

Consider $L = \operatorname{coker}\left(A \xrightarrow{(-u,v)} M \oplus N\right) \xleftarrow{\omega} A$

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "v"'] \arrow[dr, "\omega"] & M \arrow[d] \\
N \arrow[r] & L
\end{tikzcd}$$

with $T^n(\omega) = 0$.

从下证明 $f_{A,u}^n = f_{A,\omega}^n \left(= f_{A,v}^n\right)$, 从而$\overset{f_{A,u}^n}{\vee}$与 u 选取无关.

The morphism $M \to L$ induces a morphism $M/A \longrightarrow L/A$ together with

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "u"] \arrow[d, equal] & M \arrow[r] \arrow[d] & M/A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "\omega"] & L \arrow[r] & L/A \arrow[r] & 0
\end{tikzcd}$$

It induces a commutative diagram

(*)

$$\begin{tikzcd}
T^{n-1}(M) \arrow[rr] \arrow[dr, "f_M^{n-1}"] \arrow[dd] & & T^{n-1}(M/A) \arrow[rr] \arrow[dr, "f_{M/A}^{n-1}"] \arrow[dd, dashed] & & T^n(A) \arrow[r] \arrow[dr, "f_{A,u}^n"] \arrow[dd, dashed] & 0 \\
& T'^{n-1}(M) \arrow[rr] \arrow[dd] & & T'^{n-1}(M/A) \arrow[rr] \arrow[dd] & & T'^n(A) \arrow[dd, equal] \\
T^{n-1}(L) \arrow[rr, dashed] \arrow[dr, "f_L^{n-1}"'] & & T^{n-1}(L/A) \arrow[rr, dashed] \arrow[dr, dashed, "f_{L/A}^{n-1}"'] & & T^n(A) \arrow[r] \arrow[dr, dashed, "f_{A,w}^n"'] & 0 \\
& T'^{n-1}(L) \arrow[rr] & & T'^{n-1}(L/A) \arrow[rr] & & T'^n(A
\end{tikzcd}$$

$\Rightarrow f_{A,u}^n = f_{A,w}^n$. ▦

Now we show f_A^n is natural in A:

类似证明：对 $A \xrightarrow{\alpha} B \rightsquigarrow$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "u"] \arrow[d] & M \arrow[r] \arrow[d] & M/A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r, "v"] & N \arrow[r] & N/B \arrow[r] & 0
\end{tikzcd}$$

[illegible]有类似的(*)图 $\Rightarrow$ result.

Finally, given a short exact seq $0 \to A \to B \to C \to 0$, choose monomorphism $v: B \to M$ such that $T^n v = 0$.

Let $u = (A \to B \xrightarrow{v} M)$, then $T^n u = 0$.

$$\begin{tikzcd}
\Rightarrow & 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, "v"] & C \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] & A \arrow[r, "u"'] & M \arrow[r] & M/A \arrow[r] & 0
\end{tikzcd}$$

It gives rise to a commutative diagram similar to (*),

~~then $T^{n-1}C \xrightarrow{f^{n-1}} T'^{n}A$~~

Then
$$\begin{tikzcd}
T^{n+1}C \arrow[r, "\delta^{n+1}"] \arrow[d, "f_C^{n+1}"'] \arrow[dr, phantom, "\circlearrowright"] & T^nA \arrow[d] \\
T'^{n+1}C \arrow[r, "\delta^{n-1}"'] & T'^nA
\end{tikzcd}$$

∎

$\mathbb{R}(F\circ G) \cong \mathbb{R}F \circ \mathbb{R}G.$ $\left\{\begin{array}{l}\text{Spectral sequence}\\ \text{Distinguished triangles}\end{array}\right.$

§10 Examples of derived functors

R: commutative ring, $A, B \in \mathrm{Mod}_R$.

$$\mathrm{Ext}^i_R(A,B) = R^i\,\mathrm{Hom}_R(-,B)(A) \cong R^i\,\mathrm{Hom}_R(A,-)(B)$$

$$\mathrm{Tor}^R_i(A,B) = L_i(A\otimes_R -)(B) \cong L_i(-\otimes_R B)(A)$$

Balance results

先介绍一个抽象的结论：

Definition 10.1 Let $T = T(A_1, \cdots, A_p)$ be a left exact functor, covariant in some A_i, contravariant in some A_j. We call T right balanced if the following conditions hold:

(1) when any one of the covariant variables of T is replaced by an injective object, T becomes an exact functor in each of the remaining variables.

(2) when any one of the contravariant variables of T is replaced by projective object, T becomes an exact functor in each of the remaining variables.

The functors $\mathrm{Hom}(-,*)$ and $\mathrm{Hom}(-\otimes -,*)$ are examples right balanced functors.

Thm 10.2 If T is a right balanced functor, then the rig derived functors $R^*T(A_1,\cdots,\hat{A}_i,\cdots,A_p)(A_i)$ $\forall i=1,\cdots,p$ of T are isomorphic. (Same for left balanced functors).

Remark 10.3 $-\otimes_R *$ is left balanced.

10.4 Tor Functors

R commutative ring. Then Mod_R has enough projective objects (
R-module M, $\exists$ free R-module F with $F \twoheadrightarrow M$)

Can define left derived functors of right exact functors on Mo

$$\mathrm{Tor}_i^R(M,N) := L_i(-\otimes_R M)(N) \cong L_i(M\otimes_R -)(N)$$

$$= H^{-i}(M\otimes_R P^\bullet) = H_i(M\otimes_R P_\bullet)$$

where $P^\bullet = (\cdots \to P^{-1} \to P^0 \to 0)$ is a projective resolution of N (亦可用 flat resolution).

$P_\bullet = (\cdots \to P_1 \to P_0 \to 0)$ is a proj reso of N.

If $0 \to N' \to N \to N'' \to 0$ is an exact sequence of R-modules, then we have a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Tor}_1^R(M,N') \arrow[r] & \mathrm{Tor}_1^R(M,N) \arrow[r] & \mathrm{Tor}_1^R(M,N'') \arrow[dll] & \\
& M\otimes_R N' \arrow[r] & M\otimes_R N \arrow[r] & M\otimes_R N'' \arrow[r] & 0
\end{tikzcd}$$

We also have i-th hyper Tor : $\mathrm{Tor}_i^R : C(Mod_R) \longrightarrow Mod_R$.

Recall that $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow \mathbb{Z}/p \longrightarrow 0$ exact.

$$\Rightarrow \mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/p, B) = \begin{cases} B/pB & i=0 \\ {}_pB = \{b\in B \mid pb=0\} & i=1 \\ 0 & i\geq 2 \end{cases}$$

Proposition 10.5 For all abelian groups A and B, we have

(1) $\mathrm{Tor}_1^{\mathbb{Z}}(A,B)$ is a torsion abelian group.

(2) $\mathrm{Tor}_n^{\mathbb{Z}}(A,B) = 0$ for $n \geq 2$.

(3) $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, B) = B_{tor} = \left\{ \begin{array}{c} \text{torsion elements} \\ \text{of } B \end{array} \right\}$.

(4) If A is torsion-free, then $\mathrm{Tor}_n^{\mathbb{Z}}(A,B) = 0$ for $\forall n \neq 0$ and $\forall B \in Ab$.

proof (1)&(2) $A = \varinjlim A_i$, A_i f.g. $\mathrm{Tor}_n^{\mathbb{Z}}(A,B) = \varinjlim \mathrm{Tor}_n^{\mathbb{Z}}(A_i,B)$.

WMA : A is f.g and thus $A = \mathbb{Z}^m \oplus \mathbb{Z}/p_1 \oplus \cdots \oplus \mathbb{Z}/p_r$.

$\mathbb{Z}^m$ projective (hence flat) $\Rightarrow \mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}^m, -) = 0$ $\forall n \neq 0$

$\mathrm{Tor}_n^{\mathbb{Z}}(A,B) = \mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/p_1, B) \oplus \cdots \oplus \mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/p_r, B) \Rightarrow (1) \& (2)$.

(3) Since $\mathbb{Q}/\mathbb{Z} = \varinjlim$ finite abelian groups of $\mathbb{Q}/\mathbb{Z}$ $= \varinjlim_{p} \mathbb{Z}/p$

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, B) = \varinjlim \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p, B) = \varinjlim_{p} {}_pB = \bigcup_p \{b \in B \mid pb = 0\}$$

(4) clear. ▦

10.6 Flat resolution (Flat object is acyclic)

$F = - \otimes_R M : \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R$ right exact.

$L_iF(N) = \mathrm{Tor}_i^R(N, M)$.

N is F-acyclic $\Leftrightarrow L_iF(N) = 0 \ \forall i \neq 0$, i.e, $\mathrm{Tor}_i^R(N, M) = 0 \ \forall i \neq 0$

If M is flat (i.e, $- \otimes_R M$ exact), then $\mathrm{Tor}_i^R(-, M) = 0 \ \forall i \neq$
$\Rightarrow M$ is F-acyclic.

(See Prop 9.22)

Thus we can use flat resolutions to calculate $\mathrm{Tor}_i^R(N, M)$

For convenience, we recall the following fact (上半学期已证):

M flat R-module $\Leftrightarrow M^* = \mathrm{Hom}_{Ab}(M, \mathbb{Q}/\mathbb{Z})$ injective R-module

$\Leftrightarrow \mathrm{Tor}_n^R(N, M) = 0 \ \forall n \neq 0, \forall N \in \mathrm{Mod}_R$

$\Leftrightarrow \mathrm{Tor}_1^R(N, M) = 0 \ \forall N \in \mathrm{Mod}_R$.

$\Leftrightarrow \mathrm{Tor}_1^R(R/I, M) \ \forall$ f.g. ideal $I \subseteq R$

$\Leftrightarrow \mathrm{Tor}_1^R(N, M) = 0 \ \forall$ f.g. R-module N.

10.7. Ext functors $A, B \in \mathrm{Mod}_R$.

$\mathrm{Hom}_R(A,-) : \mathrm{Mod}_R \longrightarrow \mathrm{Ab}$ left exact.

$$\mathrm{Ext}^i_R(A,B) = R^i\,\mathrm{Hom}_R(A,-)(B) \quad i \geq 0$$
$$\cong R^i\,\mathrm{Hom}_R(-,B)(A)$$

$$\mathrm{Ext}^0_R(A,B) = \mathrm{Hom}_R(A,B).$$

We also have hyper-ext $\mathrm{Ext}^i_R(A,-) : C(\mathrm{Mod}_R) \longrightarrow \mathrm{Ab}$.

For any short exact seq $0 \to X \to Y \to Z \to 0$ in Mod_R, we have a long exact sequene

$$0 \longrightarrow \mathrm{Ext}^0_R(A,X) \longrightarrow \mathrm{Ext}^0_R(A,Y) \longrightarrow \mathrm{Ext}^0_R(A,Z) \to$$
$$\hookrightarrow \mathrm{Ext}^1_R(A,X) \longrightarrow \mathrm{Ext}^1_R(A,Y) \to \mathrm{Ext}^1_R(A,Z) \to \cdots$$

或者
$$0 \longrightarrow \mathrm{Ext}^0_R(Z,B) \longrightarrow \mathrm{Ext}^0_R(Y,B) \longrightarrow \mathrm{Ext}^0_R(X,B) \to$$
$$\hookrightarrow \mathrm{Ext}^1_R(Z,B) \longrightarrow \mathrm{Ext}^1_R(Y,B) \longrightarrow \mathrm{Ext}^1_R(X,B) \to \cdots$$

By def of injective objects, 以下条件等价:

$$M \text{ injective } R\text{-module} \Longleftrightarrow \mathrm{Ext}^i_R(N,M) = 0 \ \forall i \neq 0, \ \forall N$$
$$\Longleftrightarrow \mathrm{Ext}^1_R(N,M) = 0 \ \forall N.$$

Similar for projective objects.

Example 10.8 For $A, B \in Ab$, we show $\text{Ext}^n_{\mathbb{Z}}(A,B)=0 \ \forall n\geq 2$.

$\exists\, B \hookrightarrow I^0$ with I^0 injective.

Baer's Criterion $\Rightarrow I^0$ is divisible $\Rightarrow$ quotient $I'=I^0/B$ is also divisible $\Rightarrow I'=I^0/B$ is

$\Rightarrow 0\to B\to I^0\to I'\to 0$ gives an injective resolution of B.

$\Rightarrow \text{Ext}^n_{\mathbb{Z}}(A,B)=H^n(0\to \text{Hom}(A,I^0)\to \text{Hom}(A,I')\to 0)$

$=0$ for $n\geq 2$.

When $A=\mathbb{Z}/p$, $0\to\mathbb{Z}\xrightarrow{p}\mathbb{Z}\to\mathbb{Z}/p\to 0$ is a projective re

of $\mathbb{Z}/p$

$\Rightarrow \text{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/p,B)=H^n(0\to\text{Hom}_{\mathbb{Z}}(\mathbb{Z},B)\xrightarrow{p}\text{Hom}_{\mathbb{Z}}(\mathbb{Z},B)\to 0)$

$$=\begin{cases} {}_pB & n=0 \\ B/pB & n=1 \\ 0 & n\geq 2 \end{cases}$$

Since $\mathbb{Z}$ projective $\Rightarrow \text{Ext}^1_{\mathbb{Z}}(\mathbb{Z},B)=0$

For any f.g. abelian group $A=\mathbb{Z}^m\oplus\mathbb{Z}/p_1\oplus\cdots\oplus\mathbb{Z}/p_n$, one can calculate $\text{Ext}^*_{\mathbb{Z}}(A,B)$.

Now we calculate $\mathrm{Ext}^n_{\mathbb{Z}}(A,\mathbb{Z})$ for any torsion group A.

We have an injective resolution of $\mathbb{Z}$: $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$

injective object by divisible

$$\Rightarrow \mathrm{Ext}^n_{\mathbb{Z}}(A,\mathbb{Z}) = H^n\Big(0 \to \underset{\substack{\| \\ 0}}{\mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Q})} \to \mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Q}/\mathbb{Z}) \to 0\Big)$$

$$= \begin{cases} \mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Z}) \cancel{=0} & n=0 \\ \mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Q}/\mathbb{Z}) = A^* & n=1 \\ 0 & n \geq 2. \end{cases}$$

10.9 Ext and Extensions (Example of "obstruction")

$A, B \in \mathrm{Mod}_R$. We will show $\mathrm{Ext}^1_R(A,B) \cong \left\{\begin{matrix}\text{Extensions} \\ \text{of } A \text{ by } B\end{matrix}\right\}/\sim$.

An extension ξ of A by B is an exact sequence $0 \to B \to X \to A \to 0$.

Two extensions ξ and η are equivalent if there is a commutative diagram

$$\begin{tikzcd}
\xi : 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\cong"] & A \arrow[r] \arrow[d, equal] & 0 \\
\eta : 0 \arrow[r] & B \arrow[r] & Y \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

An extension is split if it is equivalent to $0 \to B \to A \oplus B \to A \to 0$.

Now we define $\Theta : \left\{\begin{matrix}\text{extensions} \\ \text{of } A \text{ by } B\end{matrix}\right\}/\sim \;\longrightarrow \mathrm{Ext}^1_R(A,B)$.

For $\xi: 0 \to B \to X \to A \to 0$, apply $\operatorname{Ext}_R^*(A,-)$ get an e

sequente $\operatorname{Hom}_R(A,X) \to \operatorname{Hom}_R(A,A) \xrightarrow{\partial_\xi} \operatorname{Ext}_R^1(A,B)$

$$id_A \longmapsto \partial_\xi(id_A)$$

We define $\Theta(\xi) = \partial_\xi(id_A)$.

若 $\partial_\xi(id_A) = 0$, then $\exists A \to X$ such that $\underbrace{A \to X \to A}_{id}$, thus $\xi: 0 \to$

Thus "the class $\Theta(\xi) = \partial_\xi(id_A)$ in $\operatorname{Ext}_R^1(A,B)$ is an obstruction to ξ being split", i.e., ξ split iff id_A lifts to $\operatorname{Hom}_R(A,X)$

iff $\Theta(\xi) = 0$ in $\operatorname{Ext}_R^1(A,B)$.

By naturality of ∂, if $\xi \sim \eta$, then $\partial_\xi(id_A) = \partial_\eta(id_A)$

$\Rightarrow \Theta$ is well-defined.

特别, ~~[illegible]~~

Lemma 10.10 If $\operatorname{Ext}_R^1(A,B) = 0$, then there is no obstructi

every extension of A by B is split.

Remark 10.11 也可使用 $\operatorname{Ext}_R^*(-,B)$ 定义 Θ.

$$\operatorname{Hom}_R(X,B) \longrightarrow \operatorname{Hom}_R(B,B) \xrightarrow{\partial} \operatorname{Ext}_R^1(A,B)$$

$$id_B \longmapsto \partial(id_B)$$

Thm 10.12 $\Theta : \left\{ \begin{array}{l} \text{equi. classes of} \\ \text{extensions of } A \text{ by } B \end{array} \right\} \longrightarrow \mathrm{Ext}^1_R(A,B)$ bijection.

such that $(0\to B\to A\oplus B\to A\to 0)$ sends to 0.

$\mathrm{Ext}^1_R(A,B)$ is an abelian group, lefthand side also has "Baer sum" such that the lefthand side is also an abelian group and Θ is an isomorphism between abelian groups.

Baer sum $\xi : 0\longrightarrow B\longrightarrow X\xrightarrow{f} A\longrightarrow 0$ two extensions

$\eta : 0\longrightarrow B\longrightarrow Y\xrightarrow{g} A\longrightarrow 0$ of A by B.

define $Z=\{(x,y)\in X\times Y \mid f(x)=g(y)\}$ = pull-back $X\times_A Y$.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] & B \arrow[r, equal] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B \arrow[r] & X \arrow[r, "f"] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Z contains three copies of B: $B\times 0$, $0\times B$, and the skew diagonal $\{(-b,b) \mid b\in B\}$.

$B\times 0$ and $0\times B$ are identified ~~[illegible]~~ in $Z' = \dfrac{Z}{\{(-b,b) \mid b\in B\}}$

Since $Z/_{0\times B} \simeq X$, $X/B \cong A \Rightarrow \phi : 0\to B\to Z'\to A\to 0$ is an extension of A by B.

We define $\xi+\eta :=$ equivalent class of ϕ.