PROOF OF RATNER'S THEOREM FOR SEMISIMPLE GROUPS

LEI YANG (SICHUAN UNIVERSITY)

1. Introduction

In 1975, Raghunathan proposed two fundamental conjectures on rigidity of unipotent actions on homogeneous spaces: Raghunathan's topological conjecture and measure conjecture. Ratner managed to prove these two conjectures in full generality around 1990 in several seminal papers [1990, Acta], [1990, Inventiones], [1991, Annals] and [1991, Duke].

According to Ratner's theorem on measure rigidity, for any homogeneous space G/Γ with a one-parameter unipotent subgroup $U = \{u(t) : t \in \mathbb{R}\}$, any U-invariant and ergodic probability measure μ on G/Γ is homogeneous, that is, there exists a periodic orbit Fx of some analytic subgroup $F \subset G$ such that μ is induced by the Haar measure μ_F of F. This implies the following Ratner's theorems on orbit closure and equidistritbuion: For any $x \in G/\Gamma$, the closure of the orbit Ux is a periodic orbit Fx of some analytic subgroup $F \subset G$, and the orbit Ux is equidistributed in Fx with respect μ_F in the following sense: For any compactly supported continuous function $f \in C_c(G/\Gamma)$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(u(t)x)\mathrm{d}t = \int_{Fx} f\mathrm{d}\mu_F.$$

Ratner's theorems have many important applications to number theory, and the ideas in the proof inspired several important breakthroughs in dynamical systems, such as the work by Einsiedler-Katok-Lindenstrauss on measure rigidity of higher rank diagonal actions on homogeneous spaces (Lindenstrauss' fields medal work), the work by Benoist-Quint on classifications of stationary measures of random walks on homogeneous spaces, and the work by Eskin-Mirzakhani on measure classification of $\mathrm{SL}_2(\mathbb{R})$-invariant measures on moduli spaces (Mirzakhani's fields medal work).

The goal of this short course is to understand Ratner's proof of measure rigidity for semisimple G [1990, Acta]. This is the most important and difficult case. The proof contains most key ideas and techniques.

2. Outline of the course

The course is organized as follows:

(1) In the first lecture, we will briefly go through the history around Raghunathan's conjecture and Ratner's theorems, and talk about some important applications of Ratner's theorems.
(2) In the second lecture, we will start the proof of Ratner's measure classification theorem for the case where G is semisimple. We will prove several important properties of unipotent actions.
(3) In the third lecture, we will statement the key lemma and start proving it. This will be the main part of the whole proof.

(4) In the fourth lecture, we will finish the proof of key lemma and briefly describe how to deduce the measure rigidity result from the key lemma.

We will closely follow Ratner's paper [1990, Acta]. The course is self-contained.

3. References

[1] Marina Ratner, On measure rigidity of unipotent subgroups of semisimple groups, Acta Math., 165, 229-309, 1990.

[2] Marina Ratner, Strict measure rigidity for unipotent subgroups of solvable groups, Invent. Math., 101, 449-482, 1990.

[3] Marina Ratner, On Raghunathan's Measure Conjecture, Annals Math., 134(3), 545-607, 1991.

[4] Marina Ratner, Raghunathan's Topological Conjecture and distributions of unipotent flows, Duke Math. J., 63(1), 235-280, 1991.

半单情形的 Ratner 测度分类定理的证明

杨磊 四川大学

一、背景介绍

Raghunathan 于 1975 年提出了两个关于齐性空间中幂幺子群作用刚性的基本猜想：Raghunathan 拓扑猜想和 Raghunathan 测度猜想。Ratner 在 1990 年左右发表了一系列开创性的论文【1990，Acta】【1990，Inventiones】【1991，Annals】【1991，Duke】，在最普遍的意义下完全证明了这两个猜想。如今这一系列结果被统称为 Ratner 定理，是齐性动力系统领域具有里程碑意义的理论。

根据 Ratner 测度分类定理，给定一个齐性空间以及一个幂幺子群作用，其上的任意一个关于该幂幺子群作用遍历的概率测度都是代数的：即该测度必然支撑在一个子李群的周期轨上，而且由该子李群上的 Haar 测度诱导而来。这个分类定理是整个 Ratner 定理的核心，由它可以推出 Ratner 轨道闭包定理和均匀分布定理：齐性空间上的任何一个幂幺子群生成的轨道的闭包必然是一个子李群的周期轨，而且该幂幺轨道在这个子李群的周期轨上关于该子李群的 Haar 测度均匀分布。

Ratner 定理在数论上有很多重要的应用，比如 Ratner 定理可以直接推出数论中著名的 Oppenheim 猜想。该猜想于 1986 年由 Margulis 证明，用的也是齐性动力系统的方法。另外，Ratner 定理的证明思想启发了之后动力系统领域的数项重大突破，比如：Einsiedler-Katok-Lindenstrauss 对于齐性空间中正熵高秩对角子群作用的测度分类定理（这是 Lindenstrauss 获菲尔兹奖的主要工作）；Benoist-Quint 对于齐性空间中随机游走稳定测度的分类定理；以及 Eskin-Mirzakhani 对于模空间中关于 SL(2,R)不变测度的分类定理（这是 Mirzakhani 获菲尔兹奖的主要工作）。

本次短课程的目标是详细介绍齐性空间为半单李群的商空间的 Ratner 测度分类定理的证明【1990，Acta】。这是整个 Ratner 定理证明的主体部分，包含了 Ratner 定理证明中的大部分关键想法。

二、课程安排

（1）在第一讲中，我们简单将回顾一下关于 Raghunathan 猜想和 Ratner 定理的发展历史，介绍 Ratner 定理的几个在数论中的重要应用。

（2）在第二讲中，我们将开始半单情形 Ratner 测度分类定理的证明。我们将证明幂幺作用的一些重要性质。

（3）第三讲中，我们将陈述证明中的关键引理并开始它的证明。证明关键引理是整个证明的主要部分。

（4）第四讲中，我们将完成关键引理的证明，并用关键引理完成整个分类定理的证明。

我们的证明来自于 Ratner 的论文【1990，Acta】。

三、参考文献

[1] Marina Ratner, On measure rigidity of unipotent subgroups of semisimple groups, Acta Math., 165, 229-309, 1990.

[2] Marina Ratner, Strict measure rigidity for unipotent subgroups of solvable groups, Invent. Math., 101, 449-482, 1990.

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